Optimal Contingent Delegation*

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June 4, 2021

Abstract

This paper investigates a mechanism design problem with a principal and two agents, and without contingent transfer. The principal has to make two decisions, one for each agent. But only the agents have the relevant information. In our framework, any dominant strategy incentive compatible decision rule is equivalent to contingent delegation: the delegation set offered to one agent depends on the other's report. Focusing on interval delegation, we characterize the principal's optimal mechanism under fairly general conditions. Remarkably, the optimal mechanism is completed determined by combining and modifying the solutions to a class of simple single-agent problems, where the other agent is assumed to report truthfully and choose his most preferred action. Finally, we also apply our results to the resource allocation problem in Alonso et al. (2014) and to a coordination problem within a multidivisional organization.

Keywords: Dominant strategy mechanism design, contingent delegation, group strategy-proofness

JEL Classification: D23, D82, L23, M11

*We are grateful for insightful comments from Dirk Bergemann, Tilman Börgers, Marina Halac, Johannes Hörner, Matthew Knudson, and Heng Liu. We also thank the seminar participants at Peking University, Yale, and YES at Columbia for helpful comments. Ju Hu and Xi Weng acknowledge the financial support of the NSFC (Grant Nos. 71803004 and 71973002). All remaining errors are ours.

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1 Introduction

This paper presents an analysis of a mechanism design problem with a principal (she) and two agents (he), and without contingent transfer. The principal must to make two decisions, one for each agent. But the information that is relevant for making decisions is horizontally dispersed among the agents. While each agent only cares about the decision for himself, the principal also cares about the interactions of the two decisions. This paper investigates the principal's optimal dominant strategy incentive compatible mechanism.

In our general framework, we posit that each agent's payoff only depends on his own state and the decision for him. In particular, we assume that each agent has a single-peaked preference so that he always prefers the decision that is closer to his own state. The principal's payoff function consists of three additively separable components. Two of them represent her potentially different preferences over each agent's decision and the corresponding state. The other component only depends on the two decisions for the two agents. It measures the principal's welfare from the interaction of the decisions. We assume that this component is either supermodular or submodular so that these two decisions are either complements or substitutes for the principal.

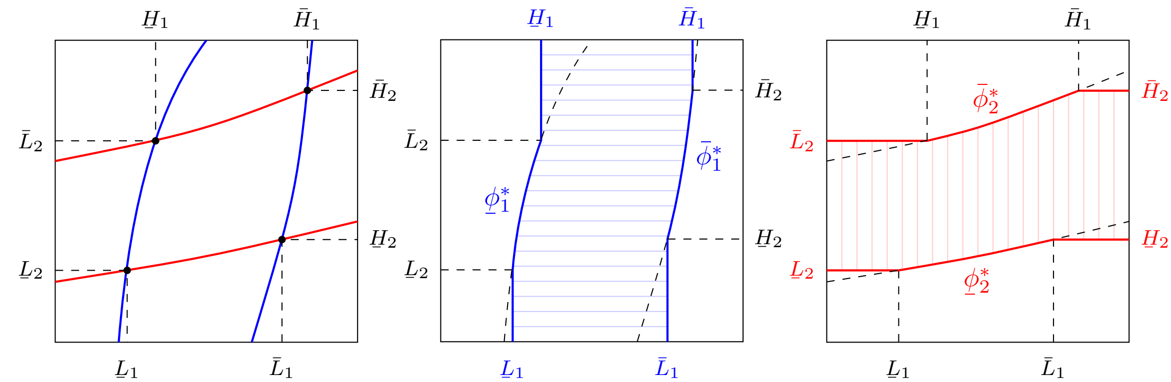
An application of our analysis is to the coordination problem within multidivisional organizations. One major reason that multidivisional organizations exist is to coordinate the decisions of their divisions. Coordinated decision making by the headquarters manager requires aggregation of the relevant information, which is usually dispersed among the individual division managers as they are best informed of their local conditions. But eliciting information from the divisions is challenging, because there is conflict of interest between the headquarters manager, who cares more about coordination, and the division managers, who cares more about adaptation: more coordinated decisions are less adapted to the local conditions of each division. Such environment has been analyzed by many studies, e.g., Alonso et al. (2008).[1] The standard approach is to compare the performance of specific mechanisms under communication equilibria. Our analysis provides a different perspective in that it determines the optimal mechanism.

In single-agent settings without contingent transfer, it is well known that the deterministic mechanism design problem is equivalent to the delegation problem in which the principal offers the agent a delegation set from which he chooses his pre-

[1] See the literature review for elaboration.

ferred one, e.g., Holmström (1977, 1984). Because we assume that each agent only cares about the decision for him and his own state in our setting, a similar equivalence holds. Any dominant strategy incentive compatible mechanism can be implemented by *contingent delegation*. In such a mechanism, the agents report their states to the principal and then the principal offers each agent a delegation set that depends on the other agent's report. After reporting and receiving his own delegation set, each agent chooses the most preferred one from it. For tractability, we restrict our attention to contingent delegation rules whose delegation sets are always intervals.[2]

Even under the interval delegation restriction, the design of the optimal contingent delegation intervals is difficult, as it is a joint optimization over four boundary functions. The main finding of this paper shows that optimal design is obtained by first decomposing the problem into a class of simple single-agent problems, which we refer to as *unilaterally constrained problems*, and then combining and modifying the solutions of these single-agent problems, which we refer to as *unilaterally constrained delegation rules*, in a certain way.



(a) Unilaterally constrained delegation rules (b) Optimal delegation for 1 (c) Optimal delegation for 2

Figure 1: Optimal delegation

More specifically, the unilaterally constrained delegation rule for agent i is the principal's optimal delegation rule for agent i when agent $-i$ is assumed to report truthfully and always choose his most preferred action. Such delegation rule is easy to obtain, as the principal only needs to choose the optimal delegation interval for agent i for every realization of agent $-i$'s state. Panel (a) in Figure 1 provides an

[2]As is the case in the single-agent setting, this restriction is equivalent to a continuity requirement in the underlying direct mechanism.

illustration of the unilaterally constrained delegation rules for both agents. The square is the s_1, s_2-plane.[3] The blue curves represent the unilaterally constrained delegation rule for agent 1. For every s_2, the interval bounded between these two curves is the delegation interval that the principal would like to grant to agent 1, assuming that agent 2 chooses his most preferred action. Similarly, the red curves represent the unilaterally constrained delegation rule for agent 2.

These unilaterally constrained delegation rules form four intersections in the s_1, s_2-plane. Then, our main result states that the optimal delegation is immediately obtained by a simple modification of the unilaterally constrained delegation rules according to these intersections. The resulting optimal delegation rules for agents 1 and 2 are illustrated in panels (b) and (c), respectively. In particular, the optimal delegation rule for agent i coincides with his unilaterally constrained delegation rule when $-i$'s state is intermediate, and remains constant when the state is extreme, i.e., when it exceeds the intersection points. As a byproduct of this structure of the optimal mechanism, truth-telling is group strategy-proof. No joint deviation by the agents can lead to a Pareto improvement.

We also provide two applications of our analysis. In the first application, we show that our analysis can be naturally adapted to the setting in Alonso et al. (2014), who study how the brain optimally allocates limited resources to different brain systems. The optimal mechanism they find takes a special form of our optimal mechanism in the sense that three of the four intersections of the unilaterally constrained delegation rules are at the boundaries of the state space. In the second application, we consider the multidivisional organization example mentioned above. Due to the simple structure of the optimal mechanism, we also obtain intuitive results on comparative statics. As one example, everything else being equal, the division whose adaptation is more important to the principal than the other one is granted larger discretion under the optimal mechanism. As another example, if one division becomes more important to the principal, then this division always benefits in that it will be granted larger discretion. But the other division will suffer as it will receive less discretion.

Related Literature. Our work relates to two main strands of literature. The first is the research on mechanism design without contingent transfers. In the single-agent setting, it is well known that such problem is equivalent to the delegation problem.

[3] For easy of exposition, we assume that the state space and the action space are the same for both agents. Our result also holds if the action space is strictly larger than the state space, since we do not need to assume the state has full support.

Holmström (1977, 1984) was the first to pose the general class of delegation problems. Since then, a number of other researches, e.g., Melumad and Shibano (1991), Martimort and Semenov (2006), Alonso and Matouschek (2008), Amador and Bagwell (2013), and Amador et al. (2018), have studied and characterized the solution to the single-agent delegation problems under various assumptions on the preferences and state distributions. This literature places particular emphasis on the optimality of interval delegation, since it is the most natural form and is commonly observed in reality. By focusing on dominant strategy incentive compatible mechanisms, we establish a similar equivalence between mechanism design and delegation in our general framework with two actions and two agents with private information. Following the tradition, we also focus on interval delegation as the first attempt to our general delegation problem with two agents.

To our knowledge, Alonso et al. (2014) were the first to study optimal mechanism design without contingent transfers in an environment with multiple actions and multiple agents.[4] In their model, a principal allocates limited resources to two agents who are privately informed of their ideal demand. They also restrict attention to dominant strategy incentive compatible mechanisms, and our result can be directly applied. In their framework, agents are biased only in one direction, in the sense that they always want to obtain more resource than the principal's ideal allocation. Therefore, only a cap will be used in the optimal unilaterally constrained delegation rules and consequently in the optimal mechanism. Our result can be considered as a generalization of theirs.[5] We point out how the decomposition result holds in the presence of biases in both directions and prove the result with fewer assumptions on the functional forms.

The second strand is studies of communication in organizations where multiple decisions must be coordinated and the relevant information for decision making is horizontally dispersed. Alonso et al. (2008), Rantakari (2008), Dessein et al. (2010), Friebel and Raith (2010), and Li and Weng (2017) explore strategic communication

[4]In a political economy model, Martimort and Semenov (2008) consider a mechanism design problem without transfer to analyze the organization of lobbying by interest groups. In their model, there are two agents, each of whom has private information about their ideal policies. However, because the policy chosen by the principal is only one-dimensional, their model is more closely related to the single-agent case.

[5]They utilize the one-sided bias to formally prove the optimality of interval delegation, while we can not provide such justification in general cases and focus on interval delegation directly. In this sense, our generalization is partial.

and the allocation of decision rights in such settings.[6] For example, Alonso et al. (2008) study a model in which the decisions of a multidivisional organization must be adapted to local conditions but also coordinated with each other. They compare the efficiency of centralization, in which case the division managers communicate vertically with the headquarters manager who will make the decisions, and decentralization, in which case the division managers who will make their own individual decisions communicate horizontally with each other. While all these papers study equilibria under certain exogenously given mechanisms, we apply our main result to this environment to investigate the optimal mechanism. To the best of our knowledge, our paper is the first to study adaptation and coordination in multidivisional organizations from the mechanism design perspective.

The rest of the paper is organized as follows. The model is presented in Section 2. In Section 3, we present our main result. Section 4 gives a sketch of the proof of our main result and explains the underlying ideas. In Section 5, we consider two applications of our main result. Section 6 concludes. All the proofs are deferred to the appendices.

2 Model Setting

There are one principal and two agents. A decision has to be made for each agent. While the principal has the legal right to make the decisions, only the agents have the necessary information to make the "right" decisions. The principal must decide on the decision-making rule. No monetary transfer is allowed.

Preferences: The principal's and the agents' payoffs depend on both the implemented decision and the state of the world. A decision consists of a pair of actions, $a_1 \in [0,1]$ for agent 1 and $a_2 \in [0,1]$ for agent 2. A state of the world is a pair $(s_1, s_2) \in [0,1]^2$, with the interpretation that s_i is agent i's state.

Agent i's payoff only depends on his own state s_i and the decision a_i for him. He has a single-peaked preference so that he always wants the decision for him to be as close to his state as possible. More specifically, his payoff function $v_i(a_i, s_i)$ satisfies (i) $v_i(\,\cdot\,, s_i)$ is maximized at $a_i = s_i$ for every s_i, and (ii) $v_i(a_i', s_i) > v_i(a_i'', s_i)$ whenever $a_i'' < a_i' < s_i$ or $a_i'' > a_i' > s_i$. A typical example is the quadratic loss payoff function $v_i(a_i, s_i) = -(a_i - s_i)^2$.

[6] There are also related models where the communication is not strategic. See, for instance, Aoki (1986), Hart and Moore (2005), Dessein and Santos (2006), and Cremer et al. (2007).

The principal, on the other hand, cares about both decisions for the two agents and their states. Her payoff function is denoted by $u_p(a_1, a_2, s_1, s_2)$. Throughout the paper, we assume that u_p takes the following additively separable form:

$$u_p(a_1, a_2, s_1, s_2) \equiv u_0(a_1, a_2) + u_1(a_1, s_1) + u_2(a_2, s_2).$$

All the components u_0, u_1 and u_2 are continuous. In addition, we make the following assumption about u_0 until Section 3.5.

Assumption P. *$u_0(a_1, a_2)$ is supermodular.*

In Section 3.5, we will discuss that our main result similarly holds if u_0 is submodular.

Information: Agent i perfectly knows his own state s_i, but not the other agent's state s_{-i}. The principal knows neither s_1 nor s_2. She believes that s_1 and s_2 are independently distributed over the interval $[0, 1]$. Denote by F_1 and F_2, respectively their cumulative distribution functions. We make the full support assumption that the supports of both F_1 and F_2 are $[0, 1]$. This is purely for ease of exposition. In the online appendix, we show that our main result can also be extended to the case where the full support assumption is not satisfied.

Because we will focus on mechanism design in dominant strategies, we do not need to specify each agent's belief about the other agent's state, e.g., common prior about the joint distribution of the states. Even the principal's prior belief can be completely subjective. It need not reflect the true distribution of the states.

Mechanism design problem: The principal has full commitment power. Thus, we can focus on direct mechanisms by the revelation principle. In a direct mechanism, the agents simultaneously report their states to the principal. Based on the two reports, the principal chooses actions for both agents. Formally, a direct mechanism consists of a pair of decision rules (a_1, a_2). Each a_i is a function that maps the reported states $(s_1, s_2) \in [0, 1]^2$ to the action $a_i(s_1, s_2) \in [0, 1]$ for agent i. For tractability, we restrict our attention to those direct mechanisms that are continuous in one's own state and dominant strategy incentive compatible.

Definition 1. A direct mechanism (a_1, a_2) is a *DSIC mechanism* if, for $i \in \{1, 2\}$, the decision rule a_i is

(i) Borel measurable;

(ii) continuous in s_i; and

(iii) dominant strategy incentive compatible: $u_i(a_i(s_i, s_{-i}); s_i) \geq u_i(a_i(\hat{s}_i, s_{-i}); s_i)$ for all s_i, $\hat{s}_i$ and s_{-i}.

Measurability is a technical condition. It guarantees that the principal's ex ante payoff is well-defined. Continuity, as we will see soon, enables us to characterize each decision rule by delegation intervals. Similar assumptions are common in the single-agent mechanism design literature such as Holmström (1984) and Armstrong (1995), as well as in the multi-agent delegation mechanism design literature such as Martimort and Semenov (2008). The notion of dominant strategy incentive compatibility is standard. It requires that reporting truthfully be always optimal regardless of the other agent's report. Dominant incentive compatibility ensures the mechanism is informationally robust (Bergemann and Morris (2005)) and it establishes a better connection with the delegation literature.

The principal's problem is then to design a DSIC mechanism to maximize her ex ante payoff:

$$\max_{(a_1, a_2)} \int_0^1 \int_0^1 u_p\left(a_1(s_1, s_2), a_2(s_1, s_2), s_1, s_2\right) \mathrm{d}F_1(s_1) \mathrm{d}F_2(s_2) \tag{1}$$
$$\text{s.t. } (a_1, a_2) \text{ is a DSIC mechanism.}$$

3 Optimal Mechanism

3.1 Contingent delegation mechanisms

For $i \in \{1, 2\}$ and $0 \leq c \leq d \leq 1$, define

$$\sigma_i(s_i; c, d) \equiv \begin{cases} c, & \text{if } s_i < c, \\ s_i, & \text{if } c \leq s_i \leq d, \\ d, & \text{if } s_i > d. \end{cases}$$

It is agent i's most preferred decision for state s_i, when he is restricted to choose from the interval $[c, d]$. The following lemma provides a simple characterization of DSIC mechanisms.

Lemma 1. *A direct (deterministic) mechanism (a_1, a_2) is a DSIC mechanism if and only if there exist Borel measurable functions $\underline{\phi}_1, \bar{\phi}_1, \underline{\phi}_2, \bar{\phi}_2 : [0, 1] \to [0, 1]$ such that, for $i \in \{1, 2\}$, we have $\underline{\phi}_i \leq \bar{\phi}_i$ and*

$$a_i(s_i, s_{-i}) = \sigma_i\left(s_i; \underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i})\right), \ \forall s_i, s_{-i}. \tag{2}$$

In single-agent settings, it is well known that the principal's direct mechanism design problem is equivalent to the delegation problem in which the principal offers the agent a delegation set, from which the agent chooses his most preferred one. Moreover, if the mechanism is continuous in the agent's state, then the corresponding delegation set is a closed interval. Lemma 1 essentially establishes a similar equivalence in our two-agent setting. Any DSIC mechanism (a_1, a_2) is equivalent to a *contingent delegation mechanism*. In such a mechanism, the agents report their states to the principal. Instead of making decisions for the agents according to (a_1, a_2), the principal then offers each agent i a delegation interval $[\underline{\phi}_i(s_{-i}),\ \bar{\phi}_i(s_{-i})]$, which is contingent on $-i$'s report and from which i is free to choose his most preferred action. In this mechanism, every agent is willing to report truthfully, because his payoff is completely determined by his own action. Equation (2) then states that the same decisions will be implemented under the DSIC mechanism and this corresponding contingent delegation mechanism.

Lemma 1 enables us to work with contingent delegation mechanisms to find an optimal one for the principal. We will follow this approach throughout the remaining analysis. The biggest advantage of doing so is that any contingent delegation mechanism is characterized by four boundary functions $\underline{\phi}_1, \bar{\phi}_1, \underline{\phi}_2, \bar{\phi}_2$. These boundary functions are all univariate and thus are easier to work with. Formally, we call a pair of functions $\phi_i = (\underline{\phi}_i, \bar{\phi}_i) : [0,1] \to [0,1]^2$ a *delegation rule* for agent i if it is Borel measurable and satisfies $\underline{\phi}_i \leq \bar{\phi}_i$. With slight abuse of terminology, we refer to a pair of delegation rules (ϕ_1, ϕ_2) as a contingent delegation mechanism, or simply a mechanism for brevity. By Lemma 1, we can reformulate the principal problem in (1) as

$$\begin{aligned} \max_{(\phi_1, \phi_2)} \int_0^1 \int_0^1 u_p \left(\sigma_1^{\phi_1}(s_1, s_2), \sigma_2^{\phi_2}(s_1, s_2), s_1, s_2\right) \mathrm{d}F_1(s_1)\mathrm{d}F_2(s_2), \\ \text{s.t. } (\phi_1, \phi_2) \text{ is a contingent delegation mechanism,} \end{aligned} \tag{3}$$

where $\sigma_i^{\phi_i}(s_i, s_{-i}) \equiv \sigma_i(s_i; \underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i}))$ for short.

3.2 Unilaterally constrained delegation rule

We now introduce a special delegation rule for each agent. The basic purpose of doing so is to decompose the principal's design problem into two single-agent delegation problems. As will be seen soon, a certain modification of these delegation rules becomes an optimal mechanism.

To fix idea, imagine the situation where agent $-i$'s state is s_{-i} and he is facing the delegation rule ϕ_{-i} such that $\underline{\phi}_{-i}(s_i) \leq s_{-i} \leq \bar{\phi}_{-i}(s_i)$ for all $s_i \in [0,1]$. In this case, regardless of agent i's state, agent $-i$'s most preferred action s_{-i} is always available to him. Hence, his optimal action choice is simply a constant $\sigma_{-i}(s_{-i}; \underline{\phi}_{-i}(s_i), \bar{\phi}_{-i}(s_i)) \equiv s_{-i}$ for all $s_i \in [0,1]$. Given agent $-i$'s state and this behavior, consider the principal's optimal interval delegation problem for agent i. We can write it as

$$\max_{0 \leq c \leq d \leq 1} \int_0^1 \left[u_0(\sigma_i(s_i; c, d), s_{-i}) + u_i(\sigma_i(s_i; c, d), s_i)\right] \mathrm{d}F_i(s_i). \tag{4}$$

By continuity of u_0 and u_i, an optimal solution to (4) always exists. Additionally, we assume that the optimal delegation interval is always unique and non-degenerate.

Assumption U. *For every $s_{-i} \in [0,1]$, there is a unique solution $(c_i^*(s_{-i}), d_i^*(s_{-i}))$ to (4). It satisfies $c_i^*(s_{-i}) < d_i^*(s_{-i})$.*

Assumptions P and U together give us two basic properties of c_i^* and d_i^*.

Lemma 2. *For $i = 1, 2$, both c_i^*, $d_i^* : [0,1] \to [0,1]$ are continuous and increasing.*

Viewing both c_i^* and d_i^* as boundary functions, (c_i^*, d_i^*) forms a delegation rule for agent i. It is indeed the principal's optimal delegation rule for agent i given that agent $-i$ is always free to choose his most preferred action. For this reason, we refer to (c_i^*, d_i^*) as the *unilaterally constrained delegation rule for agent i*. The last assumption we make is a regularity condition for the two agents' unilaterally constrained delegation rules.

Assumption R. *In the s_1, s_2-plane, the graphs of c_1^* and d_1^* intersect those of c_2^* and d_2^*, respectively, once and only once.*

Figure 2 provides an illustration of typical unilaterally constrained delegation rules that satisfy Assumption R. There are in total four intersections. We carefully label them in the graph and will follow this notation throughout the paper.

3.3 Main result

Based on the unilaterally constrained delegation rules in the previous subsection, we are now ready to state our main result. We say a mechanism (ϕ_1, ϕ_2) is increasing if all the boundary functions $\underline{\phi}_1, \bar{\phi}_1, \underline{\phi}_2$ and $\bar{\phi}_2$ are increasing. For example, the mechanism $(c_1^*, d_1^*, c_2^*, d_2^*)$ is an increasing mechanism according to Lemma 2. Let $\mathcal{M}$ be the set of all increasing mechanisms. The following theorem is our main result. It

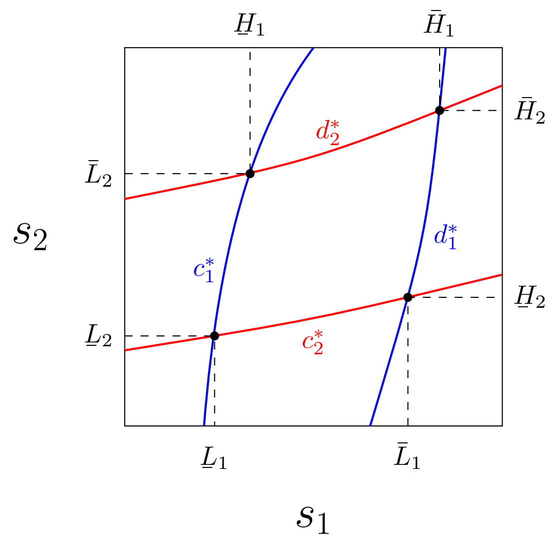


Figure 2: Unilaterally constrained delegation rules

constructs an optimal mechanism that is in $\mathcal{M}$ by modifying the unilaterally constrained delegation rules in a certain way according to their intersections. Moreover, this optimal mechanism is essentially the unique one in $\mathcal{M}$.

Theorem 1. *Suppose Assumptions P, U, and R hold. For $i = 1, 2$, define*

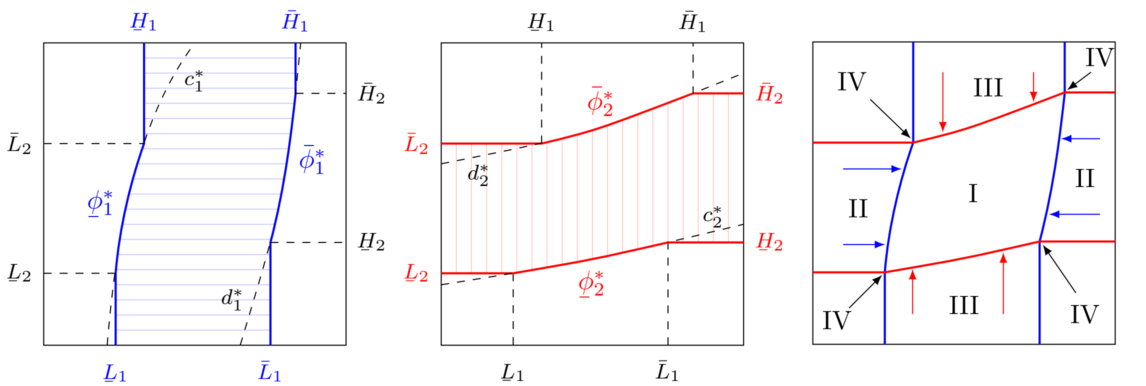
$$\underline{\phi}_i^*(s_{-i}) \equiv \begin{cases} \underline{L}_i, & \text{if } s_{-i} \in [0,\ \underline{L}_{-i}], \\ c_i^*(s_{-i}), & \text{if } s_{-i} \in (\underline{L}_{-i},\ \bar{L}_{-i}), \\ \underline{H}_i, & \text{if } s_{-i} \in [\bar{L}_{-i},\ 1], \end{cases} \tag{5}$$

and

$$\bar{\phi}_i^*(s_{-i}) \equiv \begin{cases} \bar{L}_i, & \text{if } s_{-i} \in [0,\ \underline{H}_{-i}], \\ d_i^*(s_{-i}), & \text{if } s_{-i} \in (\underline{H}_{-i},\ \bar{H}_{-i}), \\ \bar{H}_i, & \text{if } s_{-i} \in [\bar{H}_{-i},\ 1]. \end{cases} \tag{6}$$

Then, $(\phi_1^*, \phi_2^*) \in \mathcal{M}$ is an optimal mechanism. Moreover, if $(\phi_1, \phi_2) \in \mathcal{M}$ is also optimal, then $(\phi_1, \phi_2) = (\phi_1^*, \phi_2^*)$ over $(0, 1)$.

The construction of (ϕ_1^*, ϕ_2^*) can be best understood from Figure 3. Panels (a) and (b) depict the resulting delegation rules $(\underline{\phi}_1^*, \bar{\phi}_1^*)$ and $(\underline{\phi}_2^*, \bar{\phi}_2^*)$ for the two agents, respectively. Take panel (a) as an example. The blue curves represent $\underline{\phi}_1^*$ and $\bar{\phi}_1^*$. As (5) defines, $\underline{\phi}_1^*$ coincides with c_1^* when $s_2 \in (\underline{L}_2, \bar{L}_2)$. It remains constant $\underline{L}_1$ when $s_2 \in [0, \underline{L}_2]$ and constant $\underline{H}_1$ when $s_2 \in [\bar{L}_2, 1]$. Analogously, $\bar{\phi}_1^*$ coincides with d_1^* when $s_2 \in (\underline{H}_2, \bar{H}_2)$. It remains constant $\bar{L}_1$ when $s_2 \in [0, \underline{H}_2]$ and constant $\bar{H}_1$ when $s_2 \in [\bar{H}_2, 1]$.



(a) Illustration of ϕ_1^* (b) Illustration of ϕ_2^* (c) Outcomes under (ϕ_1^*, ϕ_2^*)

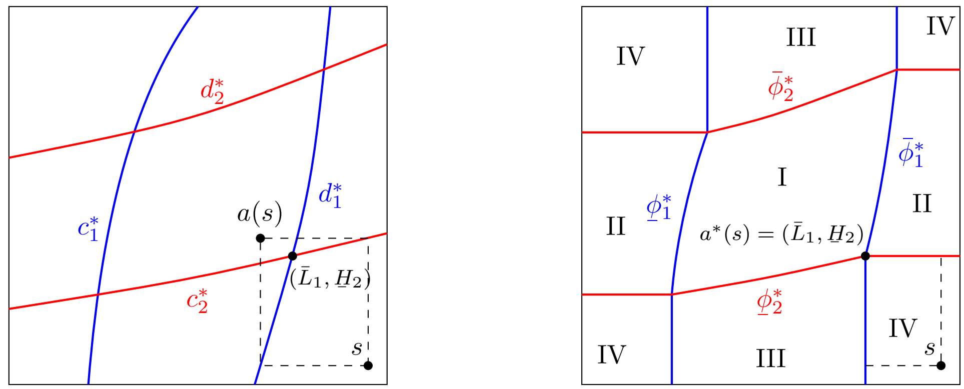
Figure 3: Optimal mechanism

Panel (c) depicts the outcome, or equivalently the corresponding DSIC mechanism, under the optimal mechanism. The arrows illustrate how a state is mapped to an action profile. The optimal mechanism divides the state space into four kinds of regions according to who is constrained. Region I is the unconstrained region in the sense that both agents are able to choose their own most preferred actions. Region II is agent 1's unilaterally constrained region. In this region, agent 2 chooses his most preferred action, but agent 1 will choose either the lower bound or the upper bound of the delegation interval for him, depending on whether his state is too low or too high. In contrast, region III is agent 2's unilaterally constrained region. At every state in this region, agent 1 chooses his own most preferred action, while agent 2 is the only one who is constrained. Lastly, region IV is the jointly constrained region. At each of these states, no one is able to choose his most preferred action.

3.4 Group Strategy-proofness

Any DSIC mechanism rules out unilateral deviations by the two agents. But in general, it does not preclude the possibility that both agents can gain by jointly misreporting. This is a particularly important concern when the principal can not completely prevent communication and thus collusion between the two agents.

As an example, consider the mechanism $(c_1^*, d_1^*, c_2^*, d_2^*)$, which consists of the unilaterally coordinated delegation rules. It is depicted in panel (a) in Figure 4. Suppose that the realized state s is around the bottom-right corner of the state space. Under this mechanism, the decision will be $a(s)$ in the graph, given that both agents report truthfully. However, if they collude and jointly misreport $\hat{s} = (\bar{L}_1, \underline{H}_2)$, then the



(a) In general, DSIC mechanism is not group strategy-proof

(b) The optimal DSIC mechanism is group strategy-proof

Figure 4: Group strategy-proofness

decision becomes $a(\hat{s}) = (\bar{L}_1, \underline{H}_2)$ and both agents become strictly better off.

In contrast, this can not occur under the optimal mechanism. Panel (b) of Figure 4 illustrates this. If both agents report truthfully, then the decision will be $a^*(s) = (\bar{L}_1, \underline{H}_2)$ under the optimal mechanism. Consider any potential joint misreport $\hat{s}$ by the two agents. The implemented decision $a^*(\hat{s})$ will be within region I (including the boundaries). We can clearly see that there is no $\hat{s}$ that can make one agent strictly better off without hurting the other. Compared to mechanism $(c_1^*, d_1^*, c_2^*, d_2^*)$, one of the key features of the optimal mechanism is that when the state s is in the jointly constrained region, i.e., region IV, the resulting decision under truthful reports is just at one of the "vertices" of region I, precluding mutual profitability of joint misreports.

Observe that if the realized state s is in the unconstrained region, i.e., region I, then both agents achieve their ideal actions under the optimal mechanism. If s is in one of the unilaterally constrained regions, i.e., region II or III, one agent achieves his ideal action under the optimal mechanism. It is easy to see that there is no other decision a within region I that does not hurt this agent but makes the other one strictly better off.

In summary, the above discussion shows that the optimal mechanism we find is in fact group strategy-proof. Hence, under the direct mechanism, the principal need not worry about collusion between the agents.

Proposition 1 (Group strategy-proofness)**.** *Let (a_1^*, a_2^*) be the corresponding DSIC mechanism of (ϕ_1^*, ϕ_2^*). For any states (s_1, s_2) and $(\hat{s}_1, \hat{s}_2)$, if $v_i(a_i^*(\hat{s}_i, \hat{s}_{-i}), s_i) > v_i(a_i^*(s_i, s_{-i}), s_i)$, then we must have $v_{-i}(a_{-i}^*(\hat{s}_i, \hat{s}_{-i}), s_{-i}) < v_{-i}(a_{-i}^*(s_i, s_{-i}), s_{-i})$.*

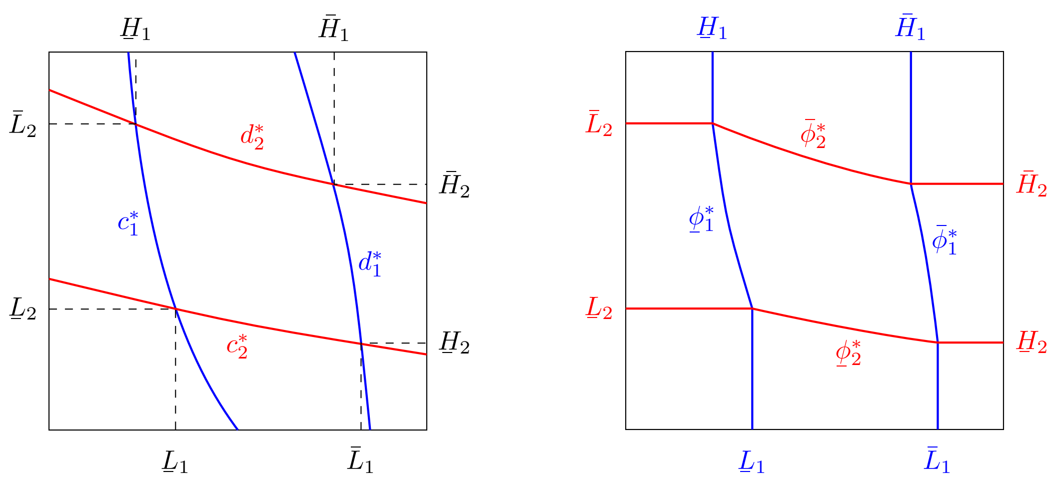
3.5 u_0 is submodular

So far, we have assumed that the component $u_0(a_1, a_2)$ in the principal's payoff function is supermodular. It is not surprising that a similar result of Theorem 1 should also hold if u_0 is submodular instead. This is because we can always relabel the state and action of agent 1 in the reverse order, and thereby transform the principal's mechanism design problem into an equivalent one where Theorem 1 can be applied. The only difference is that the optimal mechanism when u_0 is submodular will be decreasing. Formally, we make the following assumption.

Assumption P'. *$u_0(a_1, a_2)$ is submodular.*

Let $\mathcal{M}'$ be the set of all mechanisms whose boundary functions are all decreasing. We have an analogue of Theorem 1.

Theorem 1'. *Suppose Assumptions P', U and R hold. Let (ϕ_1^*, ϕ_2^*) be the mechanism defined in (5) and (6). Then, $(\phi_1^*, \phi_2^*) \in \mathcal{M}'$ is an optimal mechanism. Moreover, if $(\phi_1, \phi_2) \in \mathcal{M}'$ is also optimal, then $(\phi_1, \phi_2) = (\phi_1^*, \phi_2^*)$ over $(0, 1)$.*



(a) Unilaterally constrained delegation rules

(b) Optimal mechanism

Figure 5: Decreasing optimal mechanism for submodular u_0

Figure 5 provides an illustration of the optimal mechanism when u_0 is submodular. Panel (a) depicts the unilaterally constrained delegation rules. They are all decreasing. Panel (b) illustrates the optimal mechanism. As Theorem 1' states, the mechanism is also determined by the unilaterally constrained delegation rules and their intersections. But because u_0 is submodular, the optimal mechanism is decreasing.

4 Proof of Theorem 1

The proof of Theorem 1 is very involved. We provide a sketch in this section. Suppose that Assumptions P, U and R hold throughout.

4.1 One-sided optimal delegation

We begin with a generalization of unilaterally constrained delegation rules. It is the most important tool throughout the analysis.

Definition 2. Let $y : [0,1] \to [0,1]$ be a Borel measurable function. The pair (c,d) is called a *one-sided optimal delegation* for i given y, if

$$(c,d) \in \Gamma_i(y) \equiv \underset{0 \le \tilde{c} \le \tilde{d} \le 1}{\arg\max} \int_0^1 \Big[u_0(\sigma_i(s_i;\tilde{c},\tilde{d}), y(s_i)) + u_i(\sigma_i(s_i;\tilde{c},\tilde{d}), s_i) \Big] \, \mathrm{d}F_i(s_i). \quad (7)$$

By continuity of u_0 and u_i, $\Gamma_i(y) \neq \emptyset$ for every y. That is, at least one one-sided optimal delegation exists for every y, though it may not be unique. Observe that the pair $(c_i^*(s_{-i}),\, d_i^*(s_{-i}))$ is simply the one-sided optimal delegation for i given the constant function $y(s_i) \equiv s_{-i}$. As was motivated earlier, such constant function captures $-i$'s optimal action choice in the situation where his state is s_{-i} and he is always free to choose his most preferred action s_{-i} regardless of agent i's state. But when agent $-i$ faces a general delegation rule ϕ_{-i}, his most preferred action may not always be available to him. As a result, his optimal action choice $\sigma_{-i}(s_{-i}; \underline{\phi}_{-i}(s_i), \bar{\phi}_{-i}(s_i))$ varies with agent i's state s_i. The y function in Definition 2 is aimed to capture such situation.[7]

The following lemma points out a simple but useful property of one-sided optimal delegations. Loosely speaking, when we consider a one-sided optimal delegation (c,d) given y, the joint optimization problem in (7) can be decomposed into two separate optimization problems, one for the lower bound c and one for the upper bound d. Most importantly, c is completely determined by the lower part of y and d is completely determined by the upper part of y.

[7]The fact that this one-sided optimal delegation for i only depends on $-i$'s behavior y and is independent of his state s_{-i} is due to the additively separable form of the principal's payoff function.

Lemma 3 (Local determination)**.** *Suppose* $(c,d) \in \Gamma_i(y)$*. For any* x *such that* $c \leq x \leq d$*, we have*

$$c \in \operatorname*{arg\,max}_{0 \leq \tilde{c} \leq x} \int_0^x \left[u_0(\sigma_i(s_i;\tilde{c},x), y(s_i)) + u_i(\sigma_i(s_i;\tilde{c},x), s_i)\right] \mathrm{d}F_i(s_i), \tag{8}$$

$$d \in \operatorname*{arg\,max}_{x \leq \tilde{d} \leq 1} \int_x^1 \left[u_0(\sigma_i(s_i;x,\tilde{d}), y(s_i)) + u_i(\sigma_i(s_i;x,\tilde{d}), s_i)\right] \mathrm{d}F_i(s_i). \tag{9}$$

If, in addition, (c,d) *is unique, then both (8) and (9) hold with equality.*

For example, Assumption U and Lemma 3 together imply that, for all s_{-i} and $x \in [c_i^*(s_{-i}),\, d_i^*(s_{-i})]$, $c_i^*(s_{-i})$ is the unique solution to

$$\operatorname*{arg\,max}_{0 \leq \tilde{c} \leq x} \int_0^x \left[u_0(\sigma_i(s_i;\tilde{c},x), s_{-i}) + u_i(\sigma_i(s_i;\tilde{c},x), s_i)\right] \mathrm{d}F_i(s_i),$$

and $d_i^*(s_{-i})$ is the unique solution to

$$\operatorname*{arg\,max}_{x \leq \tilde{d} \leq 1} \int_x^1 \left[u_0(\sigma_i(s_i;x,\tilde{d}), s_{-i}) + u_i(\sigma_i(s_i;x,\tilde{d}), s_i)\right] \mathrm{d}F_i(s_i).$$

We can also naturally extend the notion of one-sided optimal delegation to mechanisms, which will give us a necessary condition for a mechanism to be optimal.

Definition 3. Consider a mechanism (ϕ_1, ϕ_2). We say ϕ_i is a *one-sided optimal delegation rule* for i given ϕ_{-i}, if, for F_{-i}−almost all s_{-i}, $(\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i}))$ is a one-sided optimal delegation for i given $\sigma_{-i}(s_{-i}; \underline{\phi}_{-i}(\,\cdot\,), \bar{\phi}_{-i}(\,\cdot\,))$. We say (ϕ_1, ϕ_2) is a pair of *mutual one-sided optimal delegation rules* if, for both $i = 1, 2$, ϕ_i is a one-sided optimal delegation for i given ϕ_{-i}.

The significance of mutual one-sided optimal delegation rules can be understood as follows. Imagine the situation where the principal has determined agent 2's delegation rule ϕ_2 and is seeking the optimal delegation rule ϕ_1 for agent 1. Notice that the component $u_2(a_2, s_2)$ in the principal's payoff function only depends on agent 2's own action and agent 2's optimal action choice under ϕ_2 is independent of ϕ_1. Thus, in considering the optimal ϕ_1, the principal only cares about her expected payoff from components $u_0(a_1, a_2)$ and $u_1(a_1, s_1)$. That is, the principal's problem is to choose ϕ_1 to maximize

$$\int_0^1 \int_0^1 \left[u_0\left(\sigma_1^{\phi_1}(s_1,s_2), \sigma_2^{\phi_2}(s_1,s_2)\right) + u_1\left(\sigma_1^{\phi_1}(s_1,s_2), s_1\right)\right] \mathrm{d}F_1(s_1)\mathrm{d}F_2(s_2).$$

Intuitively, an optimal ϕ_1 is the one that maximizes the principal's payoff for almost all realizations of s_2. That is, for almost all s_2, the pair of delegation boundaries $(\underline{\phi}_1(s_2),\, \bar{\phi}_1(s_2))$ maximizes

$$\int_0^1 \left[u_0\left(\sigma_1^{\phi_1}(s_1, s_2), \sigma_2^{\phi_2}(s_1, s_2)\right) + u_1\left(\sigma_1^{\phi_1}(s_1, s_2), s_1\right)\right] \mathrm{d}F_1(s_1).$$

In other words, ϕ_1 is an one-sided optimal delegation rule for agent 1 given ϕ_2. This discussion also leads to the following intuitive lemma. Being mutually one-sided optimal is a necessary condition for optimality.[8]

Lemma 4. *If (ϕ_1, ϕ_2) is an optimal mechanism, then it is a pair of mutual one-sided optimal delegations.*

We can now briefly discuss the main idea behind the proof of Theorem 1. Instead of directly showing that $(\phi_1^*,\, \phi_2^*)$ performs no worse than any other mechanisms, our proof takes an indirect approach. The fundamental idea of our proof is to show (i) existence — an optimal mechanism that is in $\mathcal{M}$ exists, and (ii) uniqueness — $(\bar{\phi}_1^*,\, \bar{\phi}_2^*)$ is the essentially unique pair of mutual one-sided optimal delegations in $\mathcal{M}$. These two results, together with Lemma 4, immediately imply the optimality of (ϕ_1^*, ϕ_2^*). To obtain these two results, the whole proof involves two major steps. We explain each of them in the following subsections.

4.2 Monotonicity of one-sided optimal delegations

The first step is to establish the monotonicity of one-sided optimal delegations with respect to y. It is very important for both the desired existence and uniqueness results.

Formally, let Y be the set of all Borel measurable functions from $[0, 1]$ to itself. We endow Y with the usual partial order $\geq$, where $y' \geq y$ if $y'(s) \geq y(s)$ for all $s \in [0, 1]$. Similarly, endow $\mathbb{R}^2$ with the standard product order $\geq$, where $(c', d') \geq (c, d)$ if $c' \geq c$ and $d' \geq d$. Applying the standard results on comparative statics, we obtain the following monotonicity result.

Lemma 5 (Monotonicity)**.** *For $i = 1, 2$, the one-sided optimal delegation correspondence $\Gamma_i : Y \rightrightarrows [0, 1]^2$ is increasing in the strong set order.*[9] *Moreover, there exists an increasing selection of Γ_i.*

[8] Because there is a subtlety about measurability, we defer its proof until Lemma 5.

[9] That is, if $y' \geq y$, $(c, d) \in \Gamma_i(y)$ and $(c', d') \in \Gamma_i(y')$, then $(c \wedge c', d \wedge d') \in \Gamma_i(y)$ and $(c \vee c', d \vee d') \in \Gamma_i(y')$, where $c \wedge c' \equiv \min\{c, c'\}$ and $c \vee c' \equiv \max\{c, c'\}$.

Corollary 1 below explains why Lemma 5 is useful for the desired existence result. It states that restricting attention to increasing mechanisms is without loss of generality.

Corollary 1. *For any mechanism (ϕ_1, ϕ_2), there exists an increasing mechanism $(\phi_1', \phi_2') \in \mathcal{M}$ that yields weakly higher payoff to the principal.*

By Corollary 1, any optimal mechanism *within* $\mathcal{M}$ is an optimal mechanism for the principal. Because we can show that an optimal mechanism within $\mathcal{M}$ exists, we can obtain the desired existence result.

Lemma 6 (Existence)**.** *Among all the mechanisms, there exists an optimal one in $\mathcal{M}$.*

Corollary 2 provides a basic tool for obtaining the desired uniqueness result. It is indeed frequently used in the analysis. Because it comes directly from the definition of strong set order, its proof is omitted.

Corollary 2. *Suppose $y \leq (\geq)\, y'$ and $(c, d) \in \Gamma_i(y)$.*

(i) If there exists $\hat{c}$ such that every $(c', d') \in \Gamma_i(y')$ satisfies $c' = \hat{c}$, then $c \leq (\geq)\, \hat{c}$.

(ii) If there exists $\hat{d}$ such that every $(c', d') \in \Gamma_i(y')$ satisfies $d' = \hat{d}$, then $d \leq (\geq)\, \hat{d}$.

4.3 Uniqueness of mutual one-sided optimal delegations in $\mathcal{M}$

The second step is to show that (ϕ_1^*, ϕ_2^*) is the unique pair of mutual one-sided optimal delegation rules in $\mathcal{M}$. We first derive two necessary conditions that every pair of mutual one-sided optimal delegation rules must satisfy. Based on these two conditions, we can obtain the uniqueness.

The first necessary condition provides simple global lower and upper bounds for mutual one-sided optimal delegation rules.

Lemma 7 (Global bounds)**.** *Suppose $(\phi_1, \phi_2) \in \mathcal{M}$ is a pair of mutual one-sided optimal delegation rules. For $i = 1, 2$, we have $\underline{L}_i \leq \underline{\phi}_i \leq \bar{\phi}_i \leq \bar{H}_i$ over $(0, 1)$.*

On top of lower and upper bounds, the second necessary condition provides a "bound in the middle" that separates $\underline{\phi}_i$ and $\bar{\phi}_i$ for mutual one-sided optimal delegation rules.

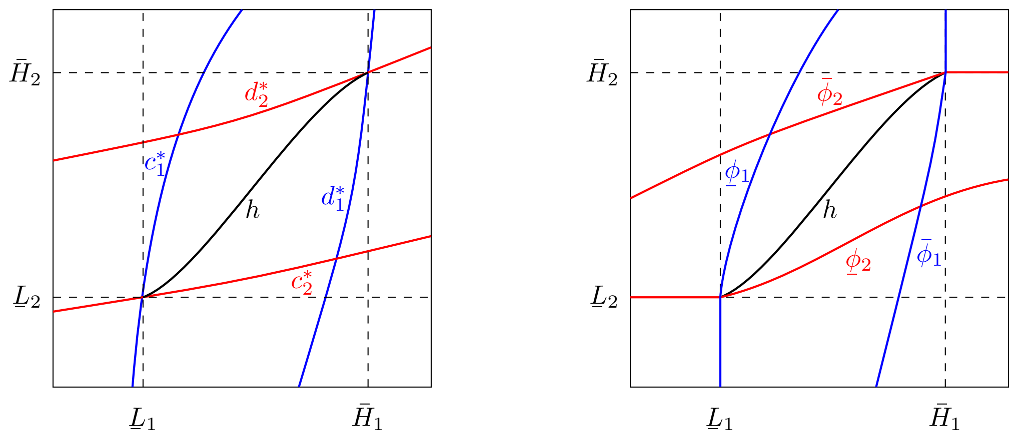
Lemma 8 (Separation)**.** *There exists a pair of mutually inverse functions h_1 and h_2 such that, for $i \in \{1, 2\}$,*

(i) $h_i : [\underline{L}_{-i}, \bar{H}_{-i}] \to [\underline{L}_i, \bar{H}_i]$ *is strictly increasing with* $h_i(\underline{L}_{-i}) = \underline{L}_i$ *and* $h_i(\bar{H}_{-i}) = \bar{H}_i$*;*

(ii) $c_i^* < h_i < d_i^*$ *over* $(\underline{L}_{-i}, \bar{H}_{-i})$*;*

and

(iii) if $(\phi_1, \phi_2) \in \mathcal{M}$ *is a pair of mutual one-sided optimal delegation rules, then* $\underline{\phi}_i \leq h_i \leq \bar{\phi}_i$ *over* $[\underline{L}_{-i}, \bar{H}_{-i}]$ *for both* $i = 1, 2$.



(a) c_i^* and d_i^* are separated by h_i

(b) $\underline{\phi}_i^*$ and $\bar{\phi}_i^*$ are separated by h_i

Figure 6: Separation property

Figure 6 provides an illustration of Lemma 8. As we can see from panel (a), it is very intuitive that we can find a strictly increasing curve (the black solid curve) that is completely contained in the interior of the area bounded by the unilaterally constrained delegation rules $(c_i^*, d_i^*)_{i=1,2}$ and that connects the two points $(\underline{L}_1, \underline{L}_2)$ and $(\bar{H}_1, \bar{H}_2)$. We denote this curve by h_2 if s_1 is its independent variable and by h_1 if s_2 is its independent variable. Thereby, h_i separates c_i^* and d_i^* over $(\underline{L}_{-i}, \bar{H}_{-i})$ in the sense that $c_i^* < h_i < d_i^*$. These are the claims of parts (i) and (ii). More importantly, any pair of mutual one-sided optimal delegation rules $(\phi_1, \phi_2) \in \mathcal{M}$ must also be separated by this curve, i.e., $\underline{\phi}_i \leq h_i \leq \bar{\phi}_i$ over $[\underline{L}_{-i}, \bar{H}_{-i}]$. This is the claim of part (iii).

Interestingly, Lemmas 7 and 8 together have already pinned down parts of any mutual one-sided optimal delegation rule $(\phi_1, \phi_2) \in \mathcal{M}$. More specifically, we must have $\underline{\phi}_i(s_{-i}) = \underline{L}_i$ for $s_{-i} \in (0, \underline{L}_{-i}]$ and $\bar{\phi}_i(s_{-i}) = \bar{H}_i$ for $s_{-i} \in [\bar{H}_{-i}, 1)$.[10] Panel (b) of Figure 6 provides an illustration.

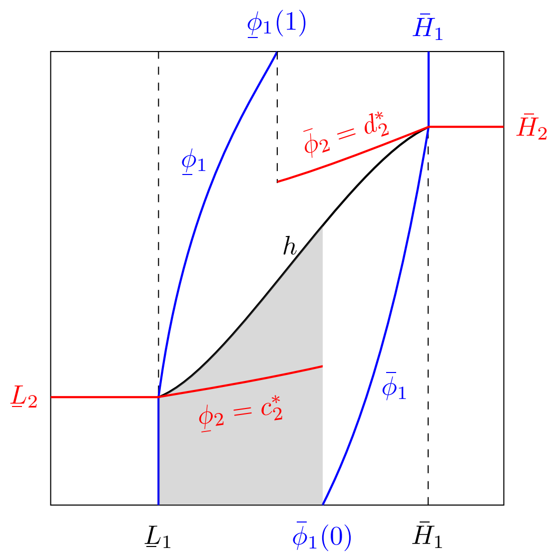


Figure 7: Uniqueness of mutual one-sided optimal delegations in $\mathcal{M}$

Furthermore, the separation property, when combined with Lemma 3, can tell us more about any pair (ϕ_1, ϕ_2) of mutual one-sided optimal delegation rules in $\mathcal{M}$. To see this, consider Figure 7. The shaded area consists of state (s_1, s_2) that satisfies $\underline{L}_1 \leq s_1 \leq \bar{\phi}_1(0)$ and $0 \leq s_2 \leq h_2(s_1)$. Because of the separation property, in particular because $\underline{\phi}_2 \leq h_2$, it is easy to observe from the graph that this shaded area is completely contained in the area bounded by $\underline{\phi}_1$ and $\bar{\phi}_1$. In other words, for any $s_1 \in [\underline{L}_1, \bar{\phi}_1(0)]$, $\sigma_1(s_1; \underline{\phi}_1(s_2), \bar{\phi}_1(s_2)) = s_1$ for all $0 \leq s_2 \leq h_2(s_1)$. Using the facts that $\underline{\phi}_2(s_1) \leq h_2(s_1) \leq \bar{\phi}_2(s_1)$ and $c_2^*(s_1) \leq h_2(s_1) \leq d_2^*(s_1)$ by the separation property and the assumption that $(c_2^*(s_1), d_2^*(s_1))$ is the unique one-sided optimal delegation rule for agent 2 given constant s_1, we immediately know from the local determination property, i.e., Lemma 3, that $\underline{\phi}_2(s_1) = c_2^*(s_1)$ for $s_1 \in [\underline{L}_1, \bar{\phi}_1(0)]$. This is illustrated by the lower red curve in the shaded area in Figure (7). Similarly, we can also show that $\bar{\phi}_2 = d_2^*$ over $[\underline{\phi}_1(1), \bar{H}_1]$. This is illustrated by the upper red curve in Figure 7.

The final task is to show that $\bar{\phi}_i(0) = \bar{L}_i$ and $\underline{\phi}_i(1) = \underline{H}_i$ for $i = 1, 2$, which will pin down the remaining parts of ϕ_1 and ϕ_2. We leave the details in the appendix. In

[10] For instance, consider $\underline{\phi}_i$. We have $\underline{L}_i \leq \underline{\phi}_i(s_{-i}) \leq \underline{\phi}_i(\underline{L}_{-i}) \leq h_i(\underline{L}_{-i}) = \underline{L}_i$, where the first inequality is from Lemma 7. The second inequality comes from monotonicity of $\underline{\phi}_i$. The third inequality comes from separation.

summary, we can obtain Lemma 9.

Lemma 9 (Uniqueness). *Suppose $(\phi_1, \phi_2) \in \mathcal{M}$ is a pair of mutual one-sided optimal delegation rules. Then, we have $(\phi_1, \phi_2) = (\phi_1^*, \phi_2^*)$ over $(0, 1)$.*

Therefore, Lemmas 4, 6, and 9 together prove Theorem 1.

5 Applications

5.1 Resource Allocation in the Brain

Our first application comes from Alonso et al. (2014). From a mechanism design approach, they analyze how the brain allocates limited resources to different brain systems that are responsible for different tasks.

Formally, there are three tasks, and system $i = 0, 1, 2$ is responsible for task i. Each system i represents a selfish entity focusing exclusively on performance in its own task. System 0 is responsible for a basic motor skill, and its needs to perform this task, s_0, is known to the principal (central executive system). Systems 1 and 2 are responsible for higher-order cognitive tasks. Their needs, s_1 and s_2, are each system's private information. The principal believes that s_1 and s_2 are independently drawn from continuous distributions with densities f_1 and f_2, respectively, over $[0, 1]$. If s_i is the actual amount of resources necessary to carry out task i flawlessly and a_i are the resources allocated to system i, the system seeks $a_i = s_i$. The principal aims to distribute a total amount of resources $k > 2$ among the three systems to maximize the overall performance of the system. Her payoff function is

$$\alpha_0 w_0(a_0 - s_0) + \alpha_1 w_1(a_1 - s_1) + \alpha_2 w_2(a_2 - s_2),$$

where $w_i(0) = w_i'(0) = 0$, $w_i'(z) > 0, u_i''(z) < 0$ for any $z < 0$, and $w(z) = 0$ for any $z > 0$. The coefficient $\alpha_i > 0$ measures the relative importance of system i. The resource constraints that the principal faces are

$$a_0 + a_1 + a_2 \leq k, \text{ and } a_i \geq 0, \ \forall i.$$

Under the shortage assumption $s_0 \geq k$ (Assumption 2 in Alonso et al. (2014)), it is easy to see that the resource allocation is always binding. Thus, we can plug the principal's resource constraints into her payoff function and reformulate it as[11]

$$u_p(a_1, a_2, s_1, s_2) = \alpha_0 w_0(k - a_1 - a_2 - s_0) + \alpha_1 w_1(a_1 - s_1) + \alpha_2 w_2(a_2 - s_2).$$

[11]For ease of exposition, we use the assumption $k > 2$ to avoid the non-negative constraint of a_0.

This specification clearly satisfies Assumption P'.

Alonso et al. (2014) further assume that the distributions of s_1 and s_2 satisfy increasing hazard rate. Under this condition, the optimal unilaterally constrained delegation rule is unique and takes the form of a contingent cap. That is, $c_i^* \equiv 0 < d_i^*$ in terms of our notation. Thus, Assumption U holds. In this case, it is also easy to see that Assumption R holds. Only d_1^* and d_2^* intersect at an interior point in the state space. The other three intersections are on the boundaries. Therefore, our Theorem 1' applies. Figure 8, which is reproduced from Figure 1 in Alonso et al. (2014), provides an illustration of the optimal mechanism.

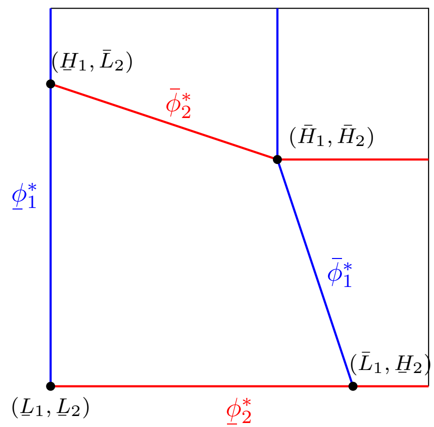


Figure 8: Optimal mechanism for resource allocation in the brain

5.2 Adaptation v.s. coordination

Our second application concerns multidivisional organizations where multiple decisions have to be coordinated but the relevant information for decision making is dispersed among the divisions. As was discussed in the literature review, some papers have studied such situations with a particular focus on communication equilibria

Alonso et al. (2014) perform careful analysis for all k. When $1 < k < 2$, one can define w_0 to be $-\infty$ when $a_0 < 0$. Although this specification violates our assumption that the objective function is continuous, it does not affect any argument in our proof. This is because continuity in our analysis is mainly to guarantee the existence of one-sided optimal delegations. In this application, the existence is guaranteed by the continuity of the payoff function when it does not take $-\infty$. Our result has to be modified if $k < 1$, because the principal is not always able to satisfy the other agent in the unilaterally constrained delegation problem, i.e., (4) is not well-defined when $s_{-i} \in (k, 1)$. In this case, we have to modify the definition of the unilaterally constrained delegation problem so that the principal satisfies the other agent's need as much as she can. Then, similar result will hold.

under exogenously given mechanisms. Here, we show that our general framework and results are ideal for studying the optimal mechanism in this situation.

Consider an organization that consists of a headquarters and two divisions. The headquarters manager is the principal, while the two division managers are the agents. As we have assumed that each agent has single-peaked preference and his payoff only depends on his own state and the decision for him, we interpret it as he only cares about his own *adaptation loss*. The principal, on the other hand, cares about both adaptation loss of the two agents and the *coordination loss*. Following Alonso et al. (2008), we measure the coordination loss of the two agents' actions by $-(a_1 - a_2)^2$ and assume that the principal's payoff function is

$$u_p(a_1, a_2, s_1, s_2) \equiv -\lambda_1(a_1 - s_1)^2 - \lambda_2(a_2 - s_2)^2 - (a_1 - a_2)^2.$$

Here, $\lambda_i > 0$ is a parameter reflecting the relative importance of agent i's adaptation loss to the principal. Clearly, this u_p satisfies Assumption P.

Furthermore, assume that the principal believes that F_i has a strictly positive and continuous density functions $f_i(s_i)$ over $(0, 1)$. We assume that both f_1 and f_2 are log-concave. The uniform distribution, which is widely used in the related literature, is an example of such distribution. More generally, the beta distribution $f_i(s_i) \propto s_i^{\alpha-1}(1 - s_i)^{\beta-1}$ for some $\alpha, \beta \geq 1$ also satisfies log-concavity. The following lemma verifies that both Assumptions U and R are satisfied with log-concave state distributions.[12]

Lemma 10. *Assumptions U and R are satisfied. Moreover,* $(\underline{L}_1, \underline{L}_2) = (0, 0)$ *and* $(\bar{H}_1, \bar{H}_2) = (1, 1)$, *and for* $i \in \{1, 2\}$, *we have* $c_i^*(s_{-i}) < s_{-i} < d_i^*(s_{-i})$ *for all* $s_{-i} \in (0, 1)$.

For a concrete example, consider the case where f_i is the uniform distribution over $[0, 1]$. We can obtain the closed form solutions for both c_i^* and d_i^*:[13]

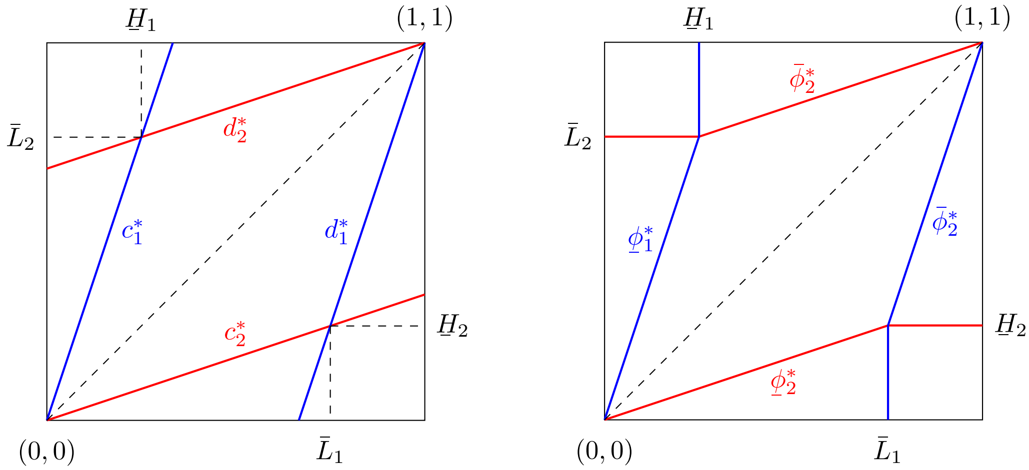
$$c_i^*(s_{-i}) = \frac{2s_{-i}}{2 + \lambda_i} \quad \text{and} \quad d_i^*(s_{-i}) = \frac{2s_{-i} + \lambda_i}{2 + \lambda_i}.$$

Panel (a) of Figure 9 illustrates these solutions. We can clearly see that both Assumptions U and R are satisfied. The unique intersection of c_1^* and c_2^* is $(0, 0)$ and that of d_1^* and d_2^* is $(1, 1)$. Moreover, c_i^* and d_i^* always lie on different sides of the diagonal, as is claimed by Lemma 10. This is intuitive, as the principal always wants

[12] All the proofs for this section are in the online appendix.

[13] Equations (C.1) and (C.2) in the online appendix provide a characterization of $c_i^*(s_{-i})$ and $d_i^*(s_{-i})$ for general log-concave density.

to make sure that agent i is able to choose the same action as agent $-i$, in which case perfect coordination is achieved.



(a) Unilaterally constrained delegation rules

(b) Optimal mechanism

Figure 9: Optimal mechanism for adaptation v.s. coordination

By Lemma 10 and Theorem 1, we immediately know the principal's optimal mechanism.

Proposition 2. *The mechanism* (ϕ_1^*, ϕ_2^*) *defined in Theorem 1 is optimal for the principal.*

Panel (b) illustrates the optimal mechanism. As is intuitive, the diagonal is completely contained in the unconstrained region. When the realized states (s_1, s_2) is on the diagonal, along which the conflict of interest between the agents and the principal vanishes, perfect adaptation and coordination are achieved simultaneously.

In the online appendix, we also provide some comparative statics results of the optimal mechanism with respect to the coefficients of relative importance and state distributions.

6 Conclusion

This paper studied optimal DSIC mechanism in an environment with two actions and two agents with private information and without monetary transfer. In this environment, any DSIC mechanism is equivalent to contingent delegation. Restricting attention to interval contingent delegation, we provide a solution to the optimal mechanism under fairly general conditions. Such optimal mechanism is simple in that it is determined by decomposing the two agents' joint delegation problem into single-agent ones assuming that the other agent is free to choose his most preferred action. We showed that our result can be applied to the existing result in the literature. We also developed a new application about adaptation v.s. coordination in multidivisional organizations. We expect to see more applications in future research.

Admittedly, the main limitation of our result is the restriction to interval contingent delegation sets. Although it is indeed without loss of generality under certain settings of our framework, as Alonso et al. (2014) have shown in their setting, unfortunately we are unable to provide a general result for our environment. As we have mentioned in the literature review, some studies of single-agent delegation settings have developed conditions for interval delegation to be optimal. But their analysis can not be easily extended to the current setting. The main difficulty comes from the fact that when determining the optimal delegation rule for agent i, the decision rule a_{-i} for the other agent will enter into the principal's objective function. In particular, even if a_{-i} is nicely behaved, there is no guarantee that interval delegation for agent i is optimal. For example, we could construct examples where a_{-i} is continuous and monotone, but it is still suboptimal to use interval delegation for agent i.[14] Finding out the conditions that guarantee the optimality of a continuous mechanism for our setting or for more general delegation problems with multiple agents remains a challenge.

Another interesting question for future research is how to find the optimal Bayesian mechanism. Although DSIC mechanism has its own conceptual advantages and makes the problem more tractable by transforming it into a contingent delegation problem, it is possible that Bayesian mechanisms can do better than the DSIC mechanisms.[15] However, due to the lack of a tractable characterization of Bayesian mechanisms, it is not clear how the optimal Bayesian mechanism can be characterized. Finally, extensions of our analysis to investigate whether stochastic mechanisms à la Kovac and Mylovanov (2009) or transfers à la Krishna and Morgan (2008) can improve the principal's expected payoff would also be interesting avenues for further work.

[14] An exception appears in our second application. In fact, we can show, by applying Proposition 1 in Amador and Bagwell (2013), that given $a_{-i} = a^*_{-i}$, i.e., the decision rule for agent $-i$ is the one under the optimal mechanism, the delegation interval $[\underline{\phi}^*_i(s_{-i}),\, \bar{\phi}^*_i(s_{-i})]$ is indeed optimal among *all delegation sets* for every s_{-i}. But clearly, this fact does not imply that (ϕ^*_1, ϕ^*_2) is optimal among all possible mechanisms.

[15] The equivalence result in Gershkov et al. (2013) does not apply due to our nonlinear setting.

Appendix A Proofs for Section 3

Proof of Lemma 1. Sufficiency is obvious. For necessity, assume (a_1, a_2) is a DSIC mechanism. For $i \in \{1, 2\}$, define

$$\underline{\phi}_i(s_{-i}) \equiv a_i(0, s_{-i}) \text{ and } \bar{\phi}_i(s_{-i}) \equiv a_i(1, s_{-i}),\ \forall s_{-i} \in [0,1].$$

Because a_i is measurable, so are $\underline{\phi}_i$ and $\bar{\phi}_i$. Because (a_1, a_2) are dominant strategy incentive compatible, a_i must be increasing in s_i. Hence, $\underline{\phi}_i(s_{-i}) \leq \bar{\phi}_i(s_{-i})$ for all $s_{-i} \in [0,1]$. Consider any $s_i < \underline{\phi}_i(s_{-i})$. By construction of $\underline{\phi}_i(s_{-i})$ and monotonicity of $a_i(\,\cdot\,, s_{-i})$, we know $s_i < a_i(0, s_{-i}) \leq a_i(s_i, s_{-i})$. Because v_i is single-peaked, we know $v_i(a_i(0, s_{-i}), s_i) \geq v_i(a_i(s_i, s_{-i}), s_i)$. But by dominant strategy incentive compatibility, we have $v_i(a_i(s_i, s_{-i}), s_i) \geq v_i(a_i(0, s_{-i}); s_i.)$. Therefore, we know $a_i(s_i, s_{-i}) = a_i(0, s_{-i}) = \underline{\phi}_i(s_{-i})$. Similarly, if $s_i > \bar{\phi}_i(s_{-i})$, we can show that $a_i(s_i, s_{-i}) = \bar{\phi}_i(s_{-i})$. Finally, consider $\underline{\phi}_i(s_{-i}) \leq s_i \leq \bar{\phi}_i(s_{-i})$. Because $a_i(\,\cdot\,, s_{-i})$ is continuous, there exists $s_i' \in [0,1]$ such that $a_i(s_i', s_{-i}) = s_i$. This implies $0 = v_i(a_i(s_i', s_{-i}), s_i) \geq v_i(a_i(s_i, s_{-i}), s_i)$. But by dominant strategy incentive compatability again, we know $v_i(a_i(s_i, s_{-i}), s_i) \geq v_i(a_i(s_i', s_{-i}), s_i)$. Therefore, we have $v_i(a_i(s_i, s_{-i}), s_i) = 0$, implying $a_i(s_i, s_{-i}) = s_i$. In summary, we have shown $a_i(s_i, s_{-i}) = \sigma_i(s_i; \underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i}))$ for all s_i and s_{-i}, completing the proof. □

Proof of Lemma 2. Continuity is standard. It comes from the maximum theorem and Assumption U. Monotonicity mainly comes from supermodularity of u_0 in Assumption P. Lemma 5 in Section 4.2 provides a more general statement, of which the current result is a direct corollary. See also Corollary 2. □

Appendix B is devoted to the proof of Theorem 1.

Proof of Proposition 1. We have explained its proof in the main text. □

Proof of Theorem 1'. Consider a new delegation problem whose primitives are $\tilde{v}_1$, $\tilde{v}_2$, $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{F}_1$, and $\tilde{F}_2$, where $\tilde{v}_1(a_1, s_1) = v_1(1 - a_1, 1 - s_1)$, $\tilde{v}_2 = v_2$, $\tilde{u}_0(a_1, a_2) = u_0(1 - a_1, a_2)$, $\tilde{u}_1(a_1, s_1) = (1 - a_1, 1 - s_1)$, $\tilde{u}_2 = u_2$, $\tilde{F}_1$ is the distribution of $1 - s_1$ under F_1, and $\tilde{F}_2 = F_2$. Because the original problem satisfies Assumptions P', U and R, it is easy to verify that the new problem satisfies Assumptions P, U and R. Moreover, there is a natural one-to-one mapping between mechanisms in the original and new problems in the sense that the corresponding mechanisms always yield the same expected payoff to the principal in these two problems respectively. If (ϕ_1, ϕ_2) is a mechanism in the original problem, then its counterpart in the new problem is

$(\tilde{\phi}_1, \tilde{\phi}_2)$, where $\tilde{\underline{\phi}}_1(s_2) = 1 - \bar{\phi}_1(s_2)$, $\bar{\tilde{\phi}}_1(s_2) = 1 - \underline{\phi}_1(s_2)$, $\tilde{\underline{\phi}}_2(s_1) = \underline{\phi}_2(1 - s_1)$, and $\bar{\tilde{\phi}}_2(s_1) = \bar{\phi}_2(1 - s_1)$. Therefore, we can apply Theorem 1 to the new problem and use this one-to-one mapping to obtain the optimal mechanism in the original problem. It is precisely the one given in Theorem 1'. □

Appendix B Proof of Theorem 1

B.1 Proofs for Section 4.1

Proof of Lemma 3. Fix $i \in \{1, 2\}$. To simplify the exposition, for every pair $0 \le c \le d \le 1$ and y, let $H_i(c, d, y)$ be the function from $[0, 1]$ to $\mathbb{R}$ defined as

$$H_i(c, d, y)(s_i) \equiv u_0 \left(\sigma_i(s_i; c, d), y(s_i)\right) + u_i \left(\sigma_i(s_i; c, d), s_i\right), \ \forall s_i \in [0, 1].$$

Hence, $\Gamma_i(y) = \arg\max_{0 \le c \le d \le 1} \int_0^1 H_i(c, d, y) \mathrm{d}F_i$.

Suppose $(c, d) \in \Gamma_i(y)$ and $c \le x \le d$. En route to a contradiction, assume at least one of (8) and (9) does not hold. Pick $c' \in \arg\max_{0 \le \tilde{c} \le x} \int_0^x H_i(\tilde{c}, x, y) \mathrm{d}F_i$ and $d' \in \arg\max_{x \le \tilde{d} \le 1} \int_{\tilde{d}}^1 H_i(x, \tilde{d}, y) \mathrm{d}F_i$. Then, we must have

$$\int_0^x H_i(c, x, y) \mathrm{d}F_i + \int_x^1 H_i(x, d, y) \mathrm{d}F_i < \int_0^x H_i(c', x, y) \mathrm{d}F_i + \int_x^1 H_i(x, d', y) \mathrm{d}F_i. \quad (10)$$

Because $c, c' \le x \le d, d'$, we can easily see that the left hand side of (10) is simply $\int_0^1 H_i(c, d, y) \mathrm{d}F_i$ and the right hand side is $\int_0^1 H_i(c', d', y) \mathrm{d}F_i$. This contradicts the assumption that $(c, d) \in \Gamma_i(y)$.

From the above argument, we can also see that any pair (c', d') that satisfies $c' \in \arg\max_{0 \le \tilde{c} \le x} \int_0^x H_i(\tilde{c}, x, y) \mathrm{d}F_i$ and $d' \in \arg\max_{x \le \tilde{d} \le 1} \int_{\tilde{d}}^1 H_i(x, \tilde{d}, y) \mathrm{d}F_i$ must also be in $\Gamma_i(y)$. Therefore, if (c, d) is unique, we must have $(c', d') = (c, d)$. □

B.2 Proofs for Section 4.2

Proof of Lemma 5. We continue to use the notation $H_i(c, d, y)$ defined in the proof of Lemma 3. Let $\pi_i(c, d, y) \equiv \int_0^1 H_i(c, d, y)(s_i) \mathrm{d}F_i(s_i)$. By Theorem 2.8.3 in Topkis (1998), to show monotonicity of Γ_i, we only need to verify that (i) for every y, π_i is supermodular in (c, d), and (ii) π_i has increasing differences in $((c, d), y)$.

Fix y and consider any (c, d) and (c', d'). Without loss of generality, assume $d \le d'$. If $c \le c'$, we clearly have $\pi(c, d, y) + \pi(c', d', y) = \pi(c \vee c', d \vee d', y) + \pi(c \wedge c', d \wedge d', y)$.

Assume $c > c'$. We thus have $c' < c \le d \le d'$. For any s_i, we can see

$$\begin{aligned}
&H_i(c', d', y)(s_i) - H_i(c \wedge c', d \wedge d', y)(s_i)\\
=&H_i(c', d', y)(s_i) - H_i(c', d, y)(s_i)\\
=&\begin{cases} 0, & \text{if } s_i \le d,\\ H_i(c, d', y)(s_i) - H_i(c, d, y)(s_i), & \text{if } s_i > d, \end{cases}\\
=&H_i(c, d', y)(s_i) - H_i(c, d, y)(s_i)\\
=&H_i(c \vee c', d \vee d', y)(s_i) - H_i(c, d, y)(s_i).
\end{aligned}$$

Therefore, $\pi_i(c, d, y) + \pi_i(c', d', y) = \pi_i(c \vee c', d \vee d', y) + \pi_i(c \wedge c', d \wedge d', y)$, implying that π_i is supermodular (and submodular) in (c, d) for every y.

Next, consider $(c', d') \ge (c, d)$. For any y, we can easily calculate

$$\begin{aligned}
&H_i(c', d', y)(s_i) - H_i(c, d, y)(s_i)\\
=&u_0(\sigma_i(s_i; c', d'), y(s_i)) - u_0(\sigma_i(s_i; c, d), y(s_i)) + \Delta,
\end{aligned}$$

where $\Delta = u_i(\sigma_i(s_i; c', d'), s_i) - u_i(\sigma_i(s_i; c, d), s_i)$ is independent of y. Because $(c', d') \ge (c, d)$, we know $\sigma_i(s_i; c', d') \ge \sigma_i(s_i; c, d)$. Hence, by the supermodularity of u_0, we have, for all $y' \ge y$,

$$H_i(c', d', y')(s_i) - H_i(c, d, y')(s_i) \ge H_i(c', d', y)(s_i) - H_i(c, d, y)(s_i),\ \forall s_i.$$

Consequently, $\pi_i(c', d', y') - \pi_i(c, d, y') \ge \pi_i(c', d', y) - \pi_i(c, d, y)$, proving that π_i has increasing differences in $((c, d), y)$. □

Proof of Corollary 1. It is clear that $\sigma_2(s_2'; \underline{\phi}_2(\cdot), \bar{\phi}_2(\cdot)) \ge \sigma_2(s_2; \underline{\phi}_2(\cdot), \bar{\phi}_2(\cdot))$ whenever $s_2' > s_2$. Thus, by Lemma 5, there exists $\phi_1' = (\underline{\phi}_1', \bar{\phi}_2')$ such that (i) ϕ_1' is a one-sided optimal delegation rule for 1 given ϕ_2, and (ii) both $\underline{\phi}_1'$ and $\bar{\phi}_1'$ are increasing. Then, (ϕ_1', ϕ_2) is a mechanism and clearly it yields an ex ante expected payoff no lower than (ϕ_1, ϕ_2) to the principal.[16] Applying the same argument, we can show that there exists $\phi_2' = (\underline{\phi}_2', \bar{\phi}_2')$ such that (i) ϕ_2' is a one-sided optimal delegation rule for agent 2 given ϕ_1', and (ii) both $\underline{\phi}_2'$ and $\bar{\phi}_2'$ are increasing. Then (ϕ_1', ϕ_2') is the desired mechanism. □

We are now ready to give the missing proof of Lemma 4 in Section 4.1.

[16]Monotone functions are Borel measurable.

Proof of Lemma 4. Suppose, by contradiction, that (ϕ_1, ϕ_2) is not a pair of mutual one-sided optimal delegation rules. Without loss of generality, assume that ϕ_1 is not a one-sided optimal delegation rule for 1 given ϕ_2. Consider the ϕ_1' constructed in the proof of Corollary 1. Then it is clear that (ϕ_1', ϕ_2) yields strictly higher ex ante expected payoff than (ϕ_1, ϕ_2) to the principal. This proves that (ϕ_1, ϕ_2) is not optimal. □

Proof of Lemma 6. We follow the standard line of proof that a continuous function over a compact set attains its maximum.

Consider the probability space $([0,1]^2, \mathscr{B}[0,1]^2, \mu_1 \times \mu_2)$, where $\mathscr{B}[0,1]^2$ is the Borel measurable sets over $[0,1]^2$. Each μ_i is the probability measure induced by F_i and $\mu_1 \times \mu_2$ is the product measure. Consider the following set of four dimensional random vectors over this probability space:

$$\mathcal{N} \equiv \left\{ (\underline{\psi}_1, \bar{\psi}_1, \underline{\psi}_2, \bar{\psi}_2) : [0,1]^2 \to [0,1]^4 \;\middle|\; \begin{array}{l} \underline{\psi}_1, \bar{\psi}_1 \text{ are constant in } s_1 \text{ and increasing in } s_2; \\ \underline{\psi}_2, \bar{\psi}_2 \text{ are increasing in } s_1 \text{ and constant in } s_2; \\ \forall i,\ \underline{\psi}_i(s,s) \le \bar{\psi}_i(s,s),\ \forall s \in [0,1]. \end{array} \right\}$$

Denote a generic element in $\mathcal{N}$ by ψ. Define the distance between $\psi, \psi' \in \mathcal{N}$ as

$$\delta(\psi, \psi') \equiv \sum_{i=1}^{2} \int_0^1 \int_0^1 (|\underline{\psi}_i - \underline{\psi}_i'| + |\bar{\psi}_i - \bar{\psi}_i'|) \mathrm{d}(\mu_1 \times \mu_2).$$

As long as we regard any two random vectors ψ and ψ' as being equivalent whenever $\psi = \psi'$ a.s., δ is indeed a metric over $\mathcal{N}$.

We first show that $(\mathcal{N}, \delta)$ is compact. For this, it suffices to show that it is sequentially compact. Consider any sequence $\{\psi_n\}_n \subset \mathcal{N}$. Because of the monotonicity properties of each ψ_n, by Helly's selection theorem, there exists a pointwise convergent subsequence $\{\psi_{n_k}\}_k$ of $\{\psi_n\}_n$.[17] Let $\psi \equiv \lim_k \psi_{n_k}$. Clearly, $\psi \in \mathcal{N}$. Then, by the bounded convergence theorem, we have $\lim_k \delta(\psi_{n_k}, \psi) = 0$, proving that $(\mathcal{N}, \delta)$ is sequentially compact.

Next, we show that the mapping $\Pi : (\mathcal{N}, \delta) \to \mathbb{R}$, defined as

$$\begin{aligned}\Pi(\psi) \equiv \int_0^1 \int_0^1 \Big\{ & u_0 \left(\sigma_1(s_1; \underline{\psi}_1(s_1,s_2), \bar{\psi}_1(s_1,s_2)), \sigma_2(s_2; \underline{\psi}_2(s_1,s_2), \bar{\psi}_2(s_1,s_2)) \right) \\ & + \sum_{i=1}^{2} u_i \left(\sigma_i(s_i; \underline{\psi}_i(s_1,s_2), \bar{\psi}_i(s_1,s_2)), s_i \right) \Big\} \mathrm{d}(\mu_1 \times \mu_2),\end{aligned}$$

is continuous. For this, we only need to show that, for any $\psi \in \mathcal{N}$ and a sequence $\{\psi_n\} \subset \mathcal{N}$ converging to ψ in δ, there is a subsequence $\{\psi_{n_k}\}_k$ such that $\Pi(\psi_{n_k}) \to$

[17] See, for instance, Rudin (1976), p. 167.

$\Pi(\psi)$. Because $\lim_n \delta(\psi_n, \psi) = 0$, we know that there exists a subsequence $\{\psi_{n_k}\}_k$ that converges to ψ a.s. By the bounded convergence theorem again, we know $\Pi(\psi_{n_k}) \to \Pi(\psi)$.

Finally, as Π is a continuous function over a compact set, it attains its maximum at some $\psi \in \mathcal{N}$. Define $\phi = (\underline{\phi}_1, \bar{\phi}_1, \underline{\phi}_2, \bar{\phi}_2) : [0,1] \to [0,1]^4$ as

$$\underline{\phi}_1(s_2) \equiv \underline{\psi}_1(0, s_2),\ \ \bar{\phi}_1(s_2) \equiv \bar{\psi}_1(0, s_2),\ \ \forall s_2 \in [0,1],$$
$$\underline{\phi}_2(s_1) \equiv \underline{\psi}_2(s_1, 0),\ \ \bar{\phi}_2(s_1) \equiv \bar{\psi}_2(s_1, 0),\ \ \forall s_1 \in [0,1].$$

Clearly, $\phi \in \mathcal{M}$ and is an optimal one among all the mechanisms in $\mathcal{M}$. By Corollary 1, ϕ is also an optimal one among all mechanisms. □

B.3 Proof for Section 4.3

To prove Lemma 7, we need the following simple lemma. It is an implication of Assumption R.

Lemma 11. *Consider $i \in \{1, 2\}$.*

(i) $c_i^*(c_{-i}^*(s_i)) > s_i$ if $s_i < \underline{L}_i$ and $c_i^*(c_{-i}^*(s_i)) < s_i$ if $s_i > \underline{L}_i$.

(ii) $d_i^*(d_{-i}^*(s_i)) > s_i$ if $s_i < \bar{H}_i$ and $d_i^*(d_{-i}^*(s_i)) < s_i$ if $s_i > \bar{H}_i$.

Proof. We show part (i). Take $i = 1$ for example. It is obvious that $(s_1, c_2^*(s_1))$ is an intersection of c_1^* and c_2^* if and only if $c_1^*(c_2^*(s_1)) = s_1$. Therefore, because of continuity of c_1^* and c_2^*, $c_1^*(c_2^*(s_1)) - s_1$ must have the same sign, either positive or negative, over $[0, \underline{L}_1)$. Because $c_1^*(c_2^*(0)) \geq 0$, we know $c_1^*(c_2^*(s_1)) - s_1$ must be positive over $[0, \underline{L}_1)$. Similarly, c_2^*, $c_1^*(c_2^*(s_1)) - s_1$ must have the same sign over $(\underline{L}_1, 1]$. Because $c_1^*(c_2^*(1)) \leq 1$, we know $c_1^*(c_2^*(s_1)) - s_1$ must be negative over $(\underline{L}_1, 1]$. □

Proof of Lemma 7. For both $i = 1, 2$, we assume without loss of generality that $(\underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i}))$ is a one-sided optimal delegation for i given $\sigma_{-i}(s_{-i}; \underline{\phi}_i(\cdot), \bar{\phi}_i(\cdot))$ for $s_{-i} = 0, 1$. Otherwise, redefine $(\underline{\phi}_i(0), \bar{\phi}_i(0)) \equiv \lim_{s_{-i} \downarrow 0} (\underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i}))$ and $(\underline{\phi}_i(1), \bar{\phi}_i(1)) \equiv \lim_{s_{-i} \uparrow 1} (\underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i}))$. Because $(\underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i}))$ is a one-sided optimal delegation for i given $\sigma_{-i}(s_{-i}; \underline{\phi}_i(\cdot), \bar{\phi}_i(\cdot))$ for F_{-i}−almost all s_{-i} and F_{-i} has full support, such limits are also one-sided optimal delegations given the corresponding behavior.

Because $\bar{\phi}_2$ is increasing, we know $\sigma_2(1; \underline{\phi}_2(s_1), \bar{\phi}_2(s_1)) = \bar{\phi}_2(s_1) \leq \bar{\phi}_2(1)$. By Corollary 2, we know

$$\bar{\phi}_1(1) \leq d_1^*(\bar{\phi}_2(1)) \text{ and } \bar{\phi}_2(1) \leq d_2^*(\bar{\phi}_1(1)).$$

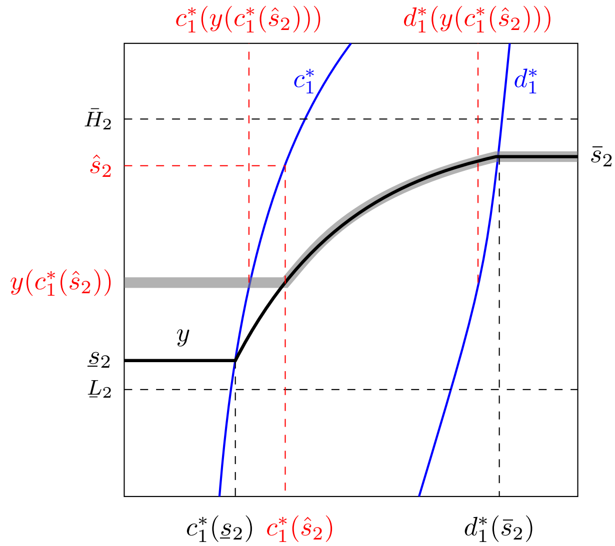


Figure 10: Proof of Lemma 12

Combining these two inequalities, we obtain

$$\bar{\phi}_1(1) \le d_1^*(d_2^*(\bar{\phi}_1(1))). \tag{11}$$

By Lemma 11, we know $\bar{\phi}_1(1) \le \bar{H}_1$, which in turn implies $\bar{\phi}_1 \le \bar{H}_1$ by monotonicity of $\bar{\phi}_1$. Similarly, we have $\bar{\phi}_2 \le \bar{H}_2$.

The other inequalities $\underline{\phi}_i \ge \underline{L}_i$ for $i = 1, 2$ can be proved analogously. □

To prove Lemma 8, we need the following lemma. It is a strong implication of Assumption U. It is also needed in the proof of Lemma 9.

Lemma 12. *Consider $i \in \{1, 2\}$. Suppose $\underline{L}_{-i} \le \underline{s}_{-i} \le \bar{s}_{-i} \le \bar{H}_{-i}$. Let $y(s_i)$ be an increasing function that satisfies*

$$y(s_i) = \begin{cases} \underline{s}_{-i}, & \text{if } s_i \in [0,\, c_i^*(\underline{s}_{-i})], \\ \bar{s}_{-i}, & \text{if } s_i \in [d_i^*(\bar{s}_{-i}),\, 1], \end{cases} \tag{12}$$

and

$$c_i^*(y(s_i)) < s_i < d_i^*(y(s_i)), \ \forall s_i \in (c_i^*(\underline{s}_{-i}),\, d_i^*(\bar{s}_{-i})). \tag{13}$$

Then the unique one-sided optimal delegation for i given y is $(c_i^*(\underline{s}_{-i}),\, d_i^*(\bar{s}_{-i}))$.

Proof. Consider $i = 1$. We show that the optimal lower bound must be $c_1^*(\underline{s}_2)$. The proof for the upper bound is similar. Define

$$S \equiv \{s_2 \in [\underline{s}_2, \bar{s}_2] \,|\, \text{every } (c,d) \in \Gamma_1(\max\{s_2, y(s_1)\}) \text{ satisfies } c = c_1^*(s_2)\}\,.$$

By construction of y, $\max\{\bar{s}_2, y(s_1)\} \equiv \bar{s}_2$. Because $\Gamma_1(\bar{s}_2) = \{(c_1^*(\bar{s}_2),\, d_1^*(\bar{s}_2))\}$ by Assumption U, we know $\bar{s}_2 \in S \neq \emptyset$. Let $\hat{s}_2 = \inf S$. For all $s_2 \in S$, we have $\hat{s}_2 \leq \max\{\hat{s}_2, y(s_1)\} \leq \max\{s_2, y(s_1)\}$ for all $s_1 \in [0,1]$. Thus, by Corollary 2, any $(c,d) \in \Gamma_1(\max\{\hat{s}_2, y(s_1)\})$ must satisfy $c_1^*(\hat{s}_2) \leq c \leq c_1^*(s_2)$ for any $s_2 \in S$, which implies $c = c_1^*(\hat{s}_2)$ by continuity of c_1^*. Thus, $\hat{s} \in S$.

The desired result will follow if we show $\hat{s}_2 = \underline{s}_2$. Suppose, by contradiction, that $\hat{s}_2 > \underline{s}_2$. In the remainder of the proof, we proceed to derive a contradiction. The analysis is divided into several small steps for clarity. In Figure 10, we carefully label the important quantities involved in the following analysis, which greatly facilitates understanding.

Step 1: $c_1^*(\underline{s}_2) < c_1^*(\hat{s}_2) < d_1^*(\bar{s}_2)$.

Because c_1^* is increasing, we know $c_1^*(\underline{s}_2) \leq c_1^*(\hat{s}_2)$. But we can not have $c_1^*(\underline{s}_2) = c_1^*(\hat{s}_2)$. To see this, note that $\underline{s}_2 \leq y(s_1) = \max\{\underline{s}_2,\, y(s_1)\} \leq \max\{\hat{s}_2,\, y(s_1)\}$ for all $s_1 \in [0,1]$. Then, for any $(c,d) \in \Gamma_1(y)$, Assumption U, Corollary 2 and the fact $\hat{s}_2 \in S$ together imply $c_1^*(\underline{s}_2) \leq c \leq c_1^*(\hat{s}_2)$. Consequently, equality $c_1^*(\underline{s}_2) = c_1^*(\hat{s}_2)$ would imply $\underline{s}_2 \in S$, which contradicts the definition of $\hat{s}_2$ and the assumption $\hat{s}_2 > \underline{s}_2$. Therefore, we must have $c_1^*(\underline{s}_2) < c_1^*(\hat{s}_2)$.

The other inequality comes directly from Assumption U and monotonicity of d_1^*: $c_1^*(\hat{s}_2) < d_1^*(\hat{s}_2) \leq d_1^*(\bar{s}_2)$.

Step 2: $c_1^*(y(c_1^*(\hat{s}_2))) < c_1^*(\hat{s}_2) < d_1^*(y(c_1^*(\hat{s}_2)))$.

This is immediate from Step 1 and the construction of y, i.e. (13).

Step 3: $\underline{s}_2 \leq y(c_1^*(\hat{s}_2)) < \hat{s}_2$.

For the first inequality, note that $\underline{s}_2 = y(c_1^*(\underline{s}_2)) \leq y(c_1^*(\hat{s}_2))$, where the equality comes from the construction of y and the inequality comes from monotonicity of both c_1^* and y. The second inequality is immediate from the first inequality in Step 2 and monotonicity of c_1^*.

Step 4: $(c,d) \in \Gamma_1(\max\{y(c_1^*(\hat{s}_2)),\, y(s_1)\})$ implies $c \leq c_1^*(\hat{s}_2) \leq d$.

By Step 3, we know $\max\{y(c_1^*(\hat{s}_2)),\, y(s_1)\} \leq \max\{\hat{s}_2,\, y(s_1)\}$. Because $\hat{s}_2 \in S$, we know $c \leq c_1^*(\hat{s}_2)$ by Corollary 2. On the other hand, because $\max\{y(c_1^*(\hat{s}_2)),\, y(s_1)\} \geq y(c_1^*(\hat{s}_2))$, we know $d \geq d_1^*(y(c_1^*(\hat{s}_2)))$ by Corollary 2 again. By Step 2, we know $d > c_1^*(\hat{s}_2)$.

Step 5: $y(c_1^*(\hat{s}_2)) \in S$.

Consider any $(c, d) \in \Gamma_1(\max\{y(c_1^*(\hat{s}_2)),\, y(s_1)\})$. Because y is increasing by construction, $\max\{y(c_1^*(\hat{s}_2)),\, y(s_1)\} = y(c_1^*(\hat{s}_2))$ for all $s_1 \in [0,\, c_1^*(\hat{s}_2)]$. By Step 4 and Lemma 3, we know

$$c \in \underset{0 \le \tilde{c} \le c_1^*(\hat{s}_2)}{\arg\max} \int_0^{c_1^*(\hat{s}_2)} [u_0(\sigma_1(s_1; \tilde{c}, c_1^*(\hat{s}_2)), s_1) + u_1(\sigma_1(s_1; \tilde{c}, c_1^*(\hat{s}_2)), y(c_1^*(\hat{s}_2))] \, \mathrm{d}F_1(s_1). \tag{14}$$

But by Step 2, Lemma 3 and Assumption U, we know that the unique solution to (14) is $c_1^*(y(c_1^*(\hat{s}_2)))$. (See the explanation right after Lemma 3.) Hence, $c = c_1^*(y(c_1^*(\hat{s}_2)))$, implying $y(c_1^*(\hat{s}_2)) \in S$.

The above steps 3 and 5 together contradict the definition of $\hat{s}_2$. Therefore, we must have $\hat{s}_2 = \underline{s}_2$, completing the proof. □

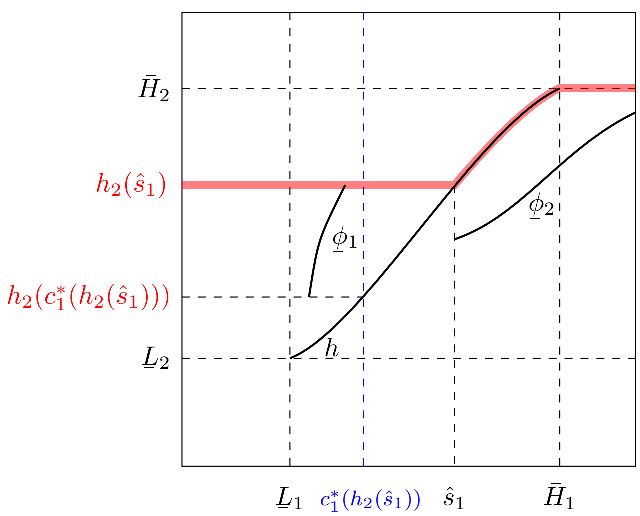


Figure 11: Proof of Lemma 8

Proof of Lemma 8. The existence of such h_1 and h_2 that satisfy parts (i) and (ii) is intuitive from panel (a) of Figure 6. We leave its formal proof to the online appendix. Here, we show that any such h_1 and h_2 must also satisfy part (iii).

Suppose (ϕ_1, ϕ_2) is a pair of mutual one-sided optimal delegation rules. Define

$$S \equiv \left\{ s_1 \in [\underline{L}_1, \bar{H}_1] \;\middle|\; \begin{array}{l} \underline{\phi}_1(s_2') \le h_1(s_2'),\ \forall s_2' \in [h_2(s_1), \bar{H}_2], \\ \underline{\phi}_2(s_1') \le h_2(s_1'),\ \forall s_1' \in [s_1, \bar{H}_1] \end{array} \right\}.$$

For $i = 1, 2$, we know $\underline{\phi}_i(\bar{H}_{-i}) \leq \bar{H}_i = h_i(\bar{H}_{-i})$, where the inequality comes from Lemma 7. Therefore, $\bar{H}_1 \in S \neq \emptyset$. Let $\hat{s}_1 \equiv \inf S$. It is easy to verify that $\hat{s}_1 \in S$. The desired result will follow if we show $\hat{s}_1 = \underline{L}_1$.

Suppose, by contradiction, $\hat{s}_1 > \underline{L}_1$. When $s_1 \in [\hat{s}_1, \bar{H}_1]$, we have $\underline{\phi}_2(s_1) \leq h_2(s_1)$. When $s_1 \in (\bar{H}_1, 1)$, we have $\underline{\phi}_2(s_1) \leq \bar{H}_2$ by Lemma 7. These two cases are illustrated in Figure 11. When $s_1 \in [0, \hat{s}_2)$, we know $\underline{\phi}_2(s_1) \leq \underline{\phi}_2(\hat{s}_1) \leq h_2(\hat{s}_1)$, where the first inequality comes from monotonicity of $\underline{\phi}_2$. In summary, for all s_1, we have

$$\underline{\phi}_2(s_1) \leq y(s_1) \equiv \begin{cases} h_2(\hat{s}_1), & \text{if } s_1 \in [0, \hat{s}_1), \\ h_2(s_1), & \text{if } s_1 \in [\hat{s}_1, \bar{H}_1], \\ \bar{H}_2, & \text{if } s_1 \in (\bar{H}_1, 1). \end{cases}$$

This y function is represented by the thick red curve in Figure 11. Consequently, for all $s_2 \in [0, h_2(\hat{s}_1)]$, we have

$$\sigma_2(s_2; \underline{\phi}_2(s_1), \bar{\phi}_2(s_1)) \leq \max\{s_2, \underline{\phi}_2(s_1)\} \leq \max\{h_2(\hat{s}_1), y(s_1)\} \leq y(s_1). \tag{15}$$

Because of parts (i) and (ii), it is easy to verify that function y satisfies conditions (12) and (13) in Lemma 12. Hence, the unique one-sided optimal delegation for 1 given y is $(c_1^*(h_2(\hat{s}_1)),\ d_1^*(\bar{H}_2))$. Because ϕ_1 is a one-sided optimal delegation rule given ϕ_2, we know that $(\underline{\phi}_1(s_2),\ \bar{\phi}_1(s_2))$ is a one-sided optimal delegation for 1 given $\sigma_2(s_2; \underline{\phi}_2(\cdot), \bar{\phi}_2(\cdot))$ for F_2-almost all $s_2 \in [0, h_2(\hat{s}_1)]$. Therefore, by (15) and Corollary 2, we know $\underline{\phi}_1(s_2) \leq c_1^*(h_2(\hat{s}_1))$ for F_2-almost all $s_2 \in [0, h_2(\hat{s}_1)]$. Because $\underline{\phi}_1$ is increasing and F_2 has full support, we actually must have $\underline{\phi}_1(s_2) \leq c_1^*(h_2(\hat{s}_1))$ for *all* $s_2 \in [0, h_2(\hat{s}_1))$. In Figure 11, this means that (the relevant part of) $\underline{\phi}_1$ is to the left of the vertical dashed blue line of value $c_1^*(h_2(\hat{s}_1))$. By part (ii), we know $c_1^*(h_2(\hat{s}_1)) < h_1(h_2(\hat{s}_1)) = \hat{s}_1$, where the equality comes from $h_1 = h_2^{-1}$. This in turn implies $h_2(c_1^*(h_2(\hat{s}_1))) < h_2(\hat{s}_1)$ since h_2 is strictly increasing, and

$$\underline{\phi}_1(s_2) \leq c_1^*(h_2(\hat{s}_1)) = h_1(h_2(c_1^*(h_2(\hat{s}_1)))) \leq h_1(s_2),\ \forall s_2 \in [h_2(c_1^*(h_2(\hat{s}_1))),\ h_2(\hat{s}_1)).$$

These inequalities can also be seen in Figure 11, as $\underline{\phi}_1$ over $[h_2(c_1^*(h_2(\hat{s}_1))),\ h_2(\hat{s}_1))$ is to the left of h_1.

Initially, we know $\underline{\phi}_1(s_2) \leq h_1(s_2)$ for all $s_2 \in [h_2(\hat{s}_1),\ \bar{H}_2]$. Now, we know $\underline{\phi}_1(s_2) \leq h_1(s_2)$ for all $s_2 \in [h_2(\hat{s}_1'),\ \bar{H}_2]$, where $\hat{s}_1' \equiv c_1^*(h_2(\hat{s}_1)) < \hat{s}_1$. Similarly, using the fact that $\underline{\phi}_1(s_2) \leq h_1(s_2)$ for all $s_2 \in [h_2(\hat{s}_1),\ \bar{H}_2]$, we can also show that there exists $\hat{s}_1'' < \hat{s}_1$ such that $\underline{\phi}_2(s_1) \leq h_2(s_1)$ for all $s_1 \in [\hat{s}_1'',\ \bar{H}_1]$. This means $\max\{\hat{s}_1',\ \hat{s}_1''\} \in S$, which contradicts the definition of $\hat{s}_1$. We therefore must have $\hat{s}_2 = \underline{L}_1$. Equivalently, for both $i = 1, 2$, $\underline{\phi}_i \leq h_i$ over $[\underline{L}_{-i},\ \bar{H}_{-i}]$.

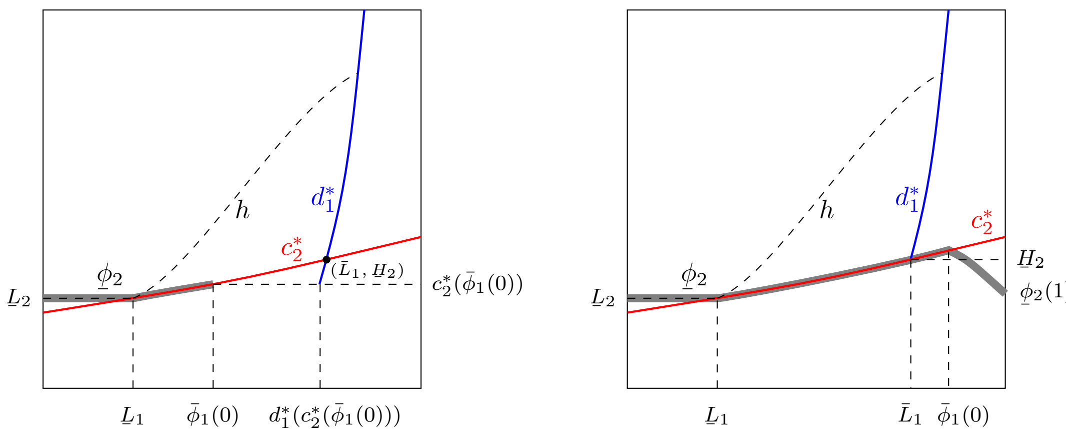
The proof of the result that $\bar{\phi}_i \geq h_i$ over $[\underline{L}_{-i},\, \bar{H}_{-i}]$ for $i = 1, 2$ is similar. $\square$

To prove Lemma 9, we need the following lemma, which is analogous to Lemma 11. Its proof is omitted.

Lemma 13. *Consider $i \in \{1, 2\}$.*

(i) $d_i^*(c_{-i}^*(s_i)) > s_i$ *if* $s_i < \bar{L}_i$ *and* $d_i^*(c_{-i}^*(s_i)) < s_i$ *if* $s_i > \bar{L}_i$.

(ii) $c_i^*(d_{-i}^*(s_i)) > s_i$ *if* $s_i < \underline{H}_i$ *and* $c_i^*(d_{-i}^*(s_i)) < s_i$ *if* $s_i > \underline{H}_i$.



(a) Illustration of Step 3 (b) Illustration of Step 4

Figure 12: Proof of Lemma 9

Proof of Lemma 9. Similarly as the proof of Lemma 7, assume $(\underline{\phi}_i(s_{-i}),\, \bar{\phi}_i(s_{-i}))$ is a one-sided optimal delegation for i given $\sigma_{-i}(s_{-i}; \underline{\phi}_{-i}(\,\cdot\,), \bar{\phi}_{-i}(\,\cdot\,))$ for both $s_{-i} = 0, 1$. Let h_1 and h_2 be the ones found in Lemma 8. The whole proof is divided into several small steps.

Step 1: For $i = 1, 2$, $\underline{\phi}_i(s_{-i}) = \underline{L}_i$ for all $s_{-i} \in (0,\, \underline{L}_{-i}]$, and $\bar{\phi}_i(s_{-i}) = \bar{H}_i$ for all $s_{-i} \in [\bar{H}_{-i},\, 1)$.

We have explained this in the main text. See Footnote 10.

Step 2: For $i = 1, 2$, $\underline{\phi}_i(s_{-i}) = c_i^*(s_{-i})$ for all $s_{-i} \in (\underline{L}_{-i},\, \bar{\phi}_{-i}(0))$, and $\bar{\phi}_i(s_{-i}) = d_i^*(s_{-i})$ for all $s_{-i} \in (\underline{\phi}_{-i}(1),\, \bar{H}_{-i})$.

Take $\underline{\phi}_2$ as an example. Consider any $s_1 \in (\underline{L}_1,\, \bar{\phi}_1(0))$ and any $s_2 \leq h_2(s_1)$. Such a pair (s_1, s_2) is a point in the shaded area in Figure 7. Note that

$$\underline{\phi}_1(s_2) \leq h_1(s_2) \leq h_1(h_2(s_1)) = s_1 < \bar{\phi}_1(0) \leq \bar{\phi}_1(s_2),$$

where the first inequality comes from Lemma 8. The second inequality comes from monotonicity of h_1. The last inequality comes from monotonicity of $\bar{\phi}_1$. This implies that, for all $s_1 \in (\underline{L}_1, \bar{\phi}_1(0))$,

$$\sigma_1(s_1; \underline{\phi}_1(s_2), \bar{\phi}_1(s_2)) = s_1, \ \forall s_2 \in (0, h_2(s_1)]. \tag{16}$$

Consider any $s_1 \in (\underline{L}_1, \bar{\phi}_1(0))$ such that $(\underline{\phi}_2(s_1), \bar{\phi}_2(s_1))$ is a one-sided optimal delegation given $\sigma_1(s_1; \underline{\phi}_1(\cdot), \bar{\phi}_2(\cdot))$. Because $\underline{\phi}_2(s_1) \leq h_2(s_1) \leq \bar{\phi}_2(s_1)$ by Lemma 8, Lemma 3 states that $\underline{\phi}_2(s_1)$ is completely determined by $\sigma_1(s_1; \underline{\phi}_1(\cdot), \bar{\phi}_1(\cdot))$ over $(0, h_2(s_1)]$, i.e.,

$$\underline{\phi}_2(s_1) \in \underset{0 \leq \tilde{c} \leq h_2(s_1)}{\arg\max} \int_0^{h_2(s_1)} [u_0(s_1, \sigma_2(s_2; \tilde{c}, h_2(s_1))) + u_2(\sigma_2(s_2; \tilde{c}, h_2(s_1)), s_2)] \, \mathrm{d}F_2(s_2). \tag{17}$$

Note that we have applied (16) in the above expression. Because $c_2^*(s_1) \leq h_2(s_2) \leq d_2^*(s_1)$ by Lemma 8, Assumption U and Lemma 3 then imply that the unique solution to the optimization problem in (17) is $c_2^*(s_1)$. Therefore, $\underline{\phi}_2(s_1) = c_2^*(s_1)$.

Because $(\underline{\phi}_2(s_1), \bar{\phi}_2(s_1))$ is a one-sided optimal delegation given $\sigma_1(s_1; \underline{\phi}_1(\cdot), \bar{\phi}_2(\cdot))$ for F_1-almost all $s_1 \in (\underline{L}_1, \bar{\phi}_1(0))$, we know from the above analysis that $\underline{\phi}_2(s_1) = c_2^*(s_1)$ for F_1-almost all $s_1 \in (\underline{L}_1, \bar{\phi}_1(0))$. Because $\underline{\phi}_2$ is increasing, c_2^* is continuous and F_1 has full support, we have $\underline{\phi}_2(s_1) = c_2^*(s_1)$ for all $s_1 \in (\underline{L}_1, \bar{\phi}_1(0))$.

Step 3: For $i = 1, 2$, we must have $\bar{\phi}_i(0) \geq \bar{L}_i$ and $\underline{\phi}_i(1) \leq \underline{H}_i$.

We take $\bar{\phi}_1(0) \geq \bar{L}_1$ as an example. Other inequalities are similar. Suppose, by contradiction, that $\bar{\phi}_1(0) < \bar{L}_1$. This situation is illustrated in panel (a) of Figure 12. The thick gray curve is $\underline{\phi}_2$. By Steps 1 and 2, we know $\underline{\phi}_2$ is constant $\underline{L}_2$ over $(0, \underline{L}_1]$ and coincides with c_2^* over $(\underline{L}_1, \bar{\phi}_1(0))$. Because $\underline{\phi}_2$ is increasing, for all $s_1 \in [\bar{\phi}_1(0), 1]$, we know

$$\underline{\phi}_2(s_1) \geq \lim_{s_1' \uparrow \bar{\phi}_1(0)} \underline{\phi}_2(s_1') = \lim_{s_1' \uparrow \bar{\phi}_1(0)} c_2^*(s_1') = c_2^*(\bar{\phi}_1(0)).$$

Therefore, we have

$$\underline{\phi}_2(s_1) \geq y(s_1) \equiv \begin{cases} \underline{L}_2, & \text{if } s_1 \in (0, \underline{L}_1], \\ c_2^*(s_1), & \text{if } s_1 \in (\underline{L}_1, \bar{\phi}_1(0)), \\ c_2^*(\bar{\phi}_1(0)), & \text{if } s_1 \in (\bar{\phi}_1(0), 1]. \end{cases}$$

This in turn implies that

$$\sigma_2(0; \underline{\phi}_2(s_1), \bar{\phi}_2(s_1)) = \underline{\phi}_2(s_1) \geq y(s_1), \ \forall s_1 \in [0, 1]. \tag{18}$$

It is easy to check that this y function satisfies conditions (12) and (13) in Lemma 12. Hence, the unique one-sided optimal delegation rule for agent 1 given y is $(\underline{L}_1,\ d_1^*(c_2^*(\bar{\phi}_1(0))))$. Because $(\underline{\phi}_1(0),\ \bar{\phi}_1(0))$ is a one-sided optimal delegation given $\sigma_2(0; \underline{\phi}_2(\,\cdot\,), \bar{\phi}_2(\,\cdot\,))$, we know $\bar{\phi}_1(0) \geq d_1^*(c_2^*(\bar{\phi}_1(0)))$ by inequality (18) and Corollary 2. By Lemma 13, we know $\bar{\phi}_1(0) \geq \bar{L}_1$, contradicting our assumption that $\bar{\phi}_1(0) < \bar{L}_1$. Therefore, we must have $\bar{\phi}_1(0) \geq \bar{L}_1$.

Step 4: For $i = 1, 2$, we must have $\bar{\phi}_i(0) = \bar{L}_i$ and $\underline{\phi}_i(1) = \underline{H}_i$.

Panel (b) of Figure 12 illustrates what would happen if $\bar{\phi}_1(0) > \bar{L}_1$ when c_2^* is strictly increasing. Again, the thick gray curve represents $\underline{\phi}_2$. By Step 2, we know $\underline{\phi}_2$ will go above $\underline{H}_2$ over $(\bar{L}_1,\ \bar{\phi}_1(0))$ as c_2^* does. But Step 3 claims that $\underline{\phi}_2(1) \leq \underline{H}_2$. Therefore, this is impossible because $\underline{\phi}_2$ is increasing.

More formally, note that the following chain of inequalities must hold

$$\bar{\phi}_1(0) \leq \bar{\phi}_1(\underline{\phi}_2(1)) \leq d_1^*(\underline{H}_2) = \bar{L}_1 \leq \bar{\phi}_1(0),$$

where the first inequality comes from monotonicity of $\bar{\phi}_1$. The second inequality comes from Steps 2 and 3. The last one comes from Step 3. Therefore, we have $\bar{\phi}_1(0) = \bar{L}_1$. The other equalities can be similarly proved.

Step 5: For $i = 1, 2$, $\underline{\phi}_i(s_{-i}) = \underline{H}_i$ for all $s_{-i} \in [\bar{L}_{-i},\ 1]$ and $\bar{\phi}_i(s_{-i}) = \bar{L}_i$ for all $s_{-i} \in [0,\ \underline{H}_{-i}]$.

This is obvious now. For example, we have

$$\underline{H}_2 = c_2^*(\bar{L}_1) \leq \underline{\phi}_2(\bar{L}_1) \leq \underline{\phi}_2(1) = \underline{H}_2,$$

where the first inequality comes from Steps 2 and 4. The second inequality comes from monotonicity of $\underline{\phi}_2$. Therefore, we have $\underline{\phi}_2(\bar{L}_1) = \underline{\phi}_2(1) = \underline{H}_2$. By monotonicity of $\underline{\phi}_2$ again, we know $\underline{\phi}_2(s_1) \equiv \underline{H}_2$ for $s_1 \in [\bar{L}_1,\ 1]$.

Combining Steps 1, 2 and 5 yields the desired result. □

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Online Appendix

This online appendix contains supplemental materials for the main text. Section A provides missing proof of Lemma 8. Section B shows that our main result also holds if F_i does not have full support over $[0, 1]$. Section C contains additional analysis on comparative statics and all the proofs for Section 5.2.

Online Appendix A Missing proof of Lemma 8

In Appendix B.3, we have proved Lemma 8 assuming that there exist desired h_1 and h_2 that satisfy parts (i) and (ii) of Lemma 8. The next lemma confirms the existence of such h_1 and h_2.

Lemma A.1. *For every $s_1 \in [\underline{L}_1, \bar{H}_1]$, there exists a unique $h_2(s_1) \in [c_2^*(s_1),\, d_2^*(s_1)]$ such that the following equation holds*

$$s_1 = \frac{h_2(s_1) - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} d_1^*(h_2(s_1)) + \frac{d_2^*(s_1) - h_2(s_1)}{d_2^*(s_1) - c_2^*(s_1)} c_1^*(h_2(s_1)). \tag{A.1}$$

Then, $h_1 \equiv h_2^{-1}$ and h_2 satisfy parts (i) and (ii) of Lemma 8.

Proof. For every $s_1 \in [\underline{L}_1, \bar{H}_1]$ and $s_2 \in [c_2^*(s_1),\, d_2^*(s_1)]$, define

$$g(s_1, s_2) \equiv \frac{s_2 - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} d_1^*(s_2) + \frac{d_2^*(s_1) - s_2}{d_2^*(s_1) - c_2^*(s_1)} c_1^*(s_2). \tag{A.2}$$

It is well defined by Assumption U and continuous by Lemma 2. We divide the remaining proof into several small steps.

Step 1: For every s_1, $g(s_1, \cdot\,)$ is strictly increasing.

Consider $c_2^*(s_1) \leq s_2 < s_2' \leq d_2^*(s_1)$. We have

$$\begin{aligned}
g(s_1, s_2) &\leq \frac{s_2 - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} d_1^*(s_2') + \frac{d_2^*(s_1) - s_2}{d_2^*(s_1) - c_2^*(s_1)} c_1^*(s_2') \\
&= \frac{s_2 - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} (d_1^*(s_2') - c_1^*(s_2')) + c_1^*(s_2') \\
&< \frac{s_2' - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} (d_1^*(s_2') - c_1^*(s_2')) + c_1^*(s_2') \\
&= g(s_1, s_2'),
\end{aligned}$$

where the first inequality comes from monotonicity of c_1^* and d_1^* by Lemma 2. The second inequality comes from $d_1^*(s_2') > c_1^*(s_2')$ by Assumption U.

Step 2: If $s_1 = \underline{L}_1$, the unique $h_2(s_1) \in [c_2^*(\underline{L}_1),\ d_2^*(\underline{L}_1)]$ that satisfies $g(s_1, h_2(s_1)) = s_1$ is $h_2(s_1) = \underline{L}_2$.

Because $c_2^*(\underline{L}_1) = \underline{L}_2$ and $c_1^*(\underline{L}_2) = \underline{L}_1$, it is straightforward to see $g(\underline{L}_1, \underline{L}_2) = \underline{L}_1$. Uniqueness comes from the previous step.

Step 3: If $s_1 = \bar{H}_1$, the unique $h_2(s_1) \in [c_2^*(\bar{H}_1),\ d_2^*(\bar{H}_1)]$ that satisfies $g(s_1, h_2(s_1)) = s_1$ is $h_2(s_1) = \bar{H}_2$.

The proof is similar to the previous one.

Step 4: If $s_1 \in (\underline{L}_1,\ \bar{H}_1)$, then there exists a unique $h_2(s_1) \in (c_2^*(s_1),\ d_2^*(s_1))$ such that $g(s_1, h_2(s_1)) = s_1$.

It is easy to see $g(s_1, c_2^*(s_1)) = c_1^*(c_2^*(s_1))$. Because $s_1 > \underline{L}_1$, we then know $g(s_1, c_2^*(s_1)) < s_1$ by Lemma 11. Similarly, because $g(s_1, d_2^*(s_1)) = d_1^*(d_2^*(s_1))$ and $s_1 < \bar{H}_1$, we know $g(s_1, d_2^*(s_1)) > s_1$ by Lemma 11 again. Thus, by Step 1, we know there exists a unique $h_2(s_1) \in (c_2^*(s_1),\ d_2^*(s_1))$ such that $g(s_1, h_2(s_1)) = s_1$.

Step 5: $h_2 : [\underline{L}_1,\ \bar{H}_1] \to [\underline{L}_2,\ \bar{H}_2]$ is continuous and surjective.

Let $\{s_1^n\}_{n\geq 1} \subset [\underline{L}_1, \bar{H}_1]$ be a sequence converging to $s_1 \in [\underline{L}_1, \bar{H}_1]$. Because $\{h_2(s_1^n)\}_{n\geq 1} \subset [\underline{L}_2, \bar{H}_2]$, it has a convergent subsequence $\{h_2(s_1^{n_k})\}_{k\geq 1}$. Let $s_2 \equiv \lim_{k\to\infty} h_2(s_1^{n_k}) \in [c_2^*(s_1),\ d_2^*(s_1)]$. Because $g(s_1^{n_k}, h_2(s_1^{n_k})) = s_1^{n_k}$ for all $k \geq 1$ and g is continuous, we know $g(s_1, s_2) = s_1$. By Steps 2 - 4, we know $s_2 = h_2(s_1)$. This proves the continuity of h_2. Because $h_2(\underline{L}_1) = \underline{L}_2$ and $h_2(\bar{H}_1) = \bar{H}_2$ by Steps 2 and 3, we know h_2 is surjective since it is continuous.

Step 6: $h_2(\underline{L}_1) < h_2(s_1) < h_2(\bar{H}_1)$ for all $s_1 \in (\underline{L}_1, \bar{H}_1)$.

For all $s_1 \in (\underline{L}_1, \bar{H}_1)$, we have

$$h_2(\underline{L}_1) = \underline{L}_2 = c_2^*(\underline{L}_1) \leq c_2^*(s_1) < h_2(s_1) < d_2^*(s_1) \leq d_2^*(\bar{H}_1) = \bar{H}_2 = h_2(\bar{H}_1),$$

where the first and last equalities come from Steps 2 and 3. The two weak inequalities come from monotonicity of c_2^* and d_2^*. The two strict inequalities come from Step 4.

Step 7: $h_2 : [\underline{L}_1, \bar{H}_1] \to [\underline{L}_2, \bar{H}_2]$ is strictly increasing.

We first argue that h_2 is injective. Consider $\underline{L}_1 \leq s_1 < s_1' \leq \bar{H}_1$. Suppose, by contradiction, $h_2(s_1) = h_2(s_1') \equiv s_2$. By Step 6, we know $\underline{L}_1 < s_1 < s_1' < \bar{H}_1$. Thus, $c_2^*(s_1) < s_2 < d_2^*(s_1)$ and $c_2^*(s_1') < s_2 < d_2^*(s_1')$ by Step 4.

Because $g(s_1, s_2) = s_1 < s_1' = g(s_1', s_2)$ and $d_1^*(s_2) > c_1^*(s_2)$, we can directly see from (A.2) that

$$\frac{s_2 - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} < \frac{s_2 - c_2^*(s_1')}{d_2^*(s_1') - c_2^*(s_1')},$$

which implies

$$\frac{d_2^*(s_1) - s_2}{s_2 - c_2^*(s_1)} > \frac{d_2^*(s_1') - s_2}{s_2 - c_2^*(s_1')}.$$

But this is impossible, since $0 < s_2 - c_2^*(s_1') \leq s_2 - c_2^*(s_1')$ and $0 < d_2^*(s_1) - s_2 \leq d_2^*(s_1') - s_2$. Therefore, h_2 is injective.

Because h_2 is continuous by Step 5, we now know h_2 is strictly monotone. Because $h_2(\underline{L}_1) < h_2(\bar{H}_1)$, we know h_2 is strictly increasing.

The above Steps 2 - 4 and 7 together guarantee that h_2 satisfies parts (i) and (ii) in Lemma 8. These steps, together with Step 5, guarantee that $h_1 \equiv h_2^{-1} : [\underline{L}_2, \bar{H}_2] \to [\underline{L}_1, \bar{H}_1]$ is well defined and satisfies part (i).

Step 8: For all $s_2 \in (\underline{L}_2, \bar{H}_2)$, $h_1(s_1) \in (c_1^*(s_2), d_1^*(s_2))$. That is, h_1 satisfies part (ii).

Let $s_1 \equiv h_1(s_2) \in (\underline{L}_1, \bar{H}_1)$. Then, (A.1) can be written as

$$h_1(s_2) = \frac{h_2(s_1) - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} d_1^*(s_2) + \frac{d_2^*(s_1) - h_2(s_1)}{d_2^*(s_1) - c_2^*(s_1)} c_1^*(s_2).$$

Because $\frac{h_2(s_1) - c_2^*(s_1)}{d_2^*(s_1) - c_2^*(s_1)} \in (0, 1)$ by Step 4, we immediately know $h_1(s_2) \in (c_1^*(s_2), d_1^*(s_2))$. This compltes the proof. $\square$

Online Appendix B Non-full-support of F_i

In the main text, we have assumed that both F_1 and F_2 have full support over $[0, 1]$. In this section, we show that the optimality of (ϕ_1^*, ϕ_2^*) will remain unchanged if this assumption is not satisfied. In fact, we provide an approach that can deal with full support or non-full-support cases in a unified manner. We begin by introducing a stronger notion of one-sided optimal delegation rule.

Definition B.1. Consider a mechanism (ϕ_1, ϕ_2). We say ϕ_i is a *pointwise one-sided optimal delegation rule* for i given ϕ_{-i}, if every $s_{-i} \in [0, 1]$, $(\underline{\phi}_i(s_{-i}), \bar{\phi}_i(s_{-i}))$ is a one-sided optimal delegation for i given $\sigma_{-i}(s_{-i}; \underline{\phi}_{-i}(\cdot), \bar{\phi}_{-i}(\cdot))$. We say (ϕ_1, ϕ_2) is a pair of *pointwise mutual one-sided optimal delegation rules* if, for both $i = 1, 2$, ϕ_i is a pointwise one-sided optimal delegation for i given ϕ_{-i}.

Clearly, the notion of pointwise one-sided optimal delegation rule is stronger than that of one-sided optimal delegation rule in Definition 3. It requires that $(\underline{\phi}_i(s_{-i}),\ \bar{\phi}_i(s_{-i}))$ be a one-sided optimal delegation for i given $\sigma_{-i}(s_{-i}; \underline{\phi}_{-i}(\cdot), \bar{\phi}_{-i}(\cdot))$ for *every* $s_{-i} \in [0,1]$, including those s_{-i}'s that are outside the support of F_{-i}.

A careful examination of the proof of Lemma 9 will reveal that the following result holds.

Lemma B.1. *Suppose $(\phi_1, \phi_2) \in \mathcal{M}$ is a pair of pointwise mutual one-sided optimal delegation rules. Then, we have $(\phi_1, \phi_2) = (\phi_1^*, \phi_2^*)$ over $(0,1)$.*

Therefore, to show the optimality of (ϕ_1^*, ϕ_2^*), it suffices to show that there exists an optimal mechanism (ϕ_1, ϕ_2) such that (i) it is in $\mathcal{M}$ and (ii) it is a pair of pointwise mutual one-sided optimal delegation rules. For this, we need one additional property of the one-sided optimal delegation correspondence.

Lemma B.2. *For any $y \in Y$, the maximum element of $\Gamma_i(y)$, denoted by $\max \Gamma_i(y)$, exists. Moreover, if $\bar{y} \geq \underline{y}$ F_i-a.s., then $\max \Gamma_i(\bar{y}) \geq \max \Gamma_i(\underline{y})$.*

Proof. It directly comes from the proof of Lemma 5 and Theorem 2.7.1 in Topkis (1998). □

We are now ready to prove the desired result.

Lemma B.3. *There exists an optimal mechanism (ϕ_1, ϕ_2) such that (i) it is in $\mathcal{M}$ and (ii) it is a pair of pointwise mutual one-sided optimal delegation rules.*

Proof. By Lemma 6, we know that there exists an optimal mechanism (ϕ_1^0, ϕ_2^0) in $\mathcal{M}$. By Lemma 4, we know it is a pair of mutual one-sided optimal delegation rules.[18] Define a sequence of mechanisms $\{(\phi_1^k, \phi_2^k)\}_{k\geq 0}$ recursively as follows: for $i = 1, 2$,

$$\phi_i^k(s_{-i}) \equiv \max \Gamma_i(\sigma_{-i}(s_{-i}; \underline{\phi}_i^{k-1}(\cdot), \bar{\phi}_i^{k-1}(\cdot))),\ \forall s_{-i} \in [0,1]. \tag{B.1}$$

We can make several simple observations. First, for every $k \geq 1$, $(\phi_1^k, \phi_2^k) \in \mathcal{M}$ and yields the same expected payoff to the principal as (ϕ_1^0, ϕ_2^0). This comes from the same line of arguments as in the proof of Corollary 1, and the assumption that (ϕ_1^0, ϕ_2^0) is optimal. Second, $\phi_i^1 \geq \phi_i^0$ F_{-i}-a.s. This is because, for F_{-i}-almost all s_{-i}, $\phi_i^0(s_{-i})$ is an element in $\Gamma_i(\sigma_{-i}(s_{-i}; \underline{\phi}_{-i}^0(\cdot), \bar{\phi}_{-i}^0(\cdot)))$, but $\phi_i^1(s_{-i})$ is the maximal element by construction. Third, $\phi_i^k \geq \phi_i^{k-1}$ pointwisely for all $k \geq 2$. Consider $k = 2$. Because $\phi_{-i}^1 \geq \phi_{-i}^0$ F_i-a.s., we know, for every s_{-i},

$$\sigma_{-i}(s_{-i}; \underline{\phi}_{-i}^1(\cdot), \bar{\phi}_{-i}^1(\cdot)) \geq \sigma_{-i}(s_{-i}; \underline{\phi}_{-i}^0(\cdot), \bar{\phi}_{-i}^0(\cdot))\ F_i\text{-a.s.}$$

[18] Note that Lemmas 4 and 6 do not rely on the original full support assumption.

Thus, by Lemma B.3, we know $\phi_i^2(s_{-i}) \geq \phi_i^1(s_{-i})$ for every s_{-i}. We then can show that $\phi_i^k \geq \phi_i^{k-1}$ for all $k > 2$ inductively.

Define (ϕ_1, ϕ_2) to be the pointwise limit of $\{(\phi_1^k, \phi_2^k)\}_{k\geq 2}$:

$$\phi_i(s_{-i}) \equiv \lim_{k\to\infty} \phi_i^k(s_{-i}),\ \forall s_{-i} \in [0,1].$$

Because we have known that $\{\phi_i^k\}_{k\geq 2}$ is a pointwise increasing sequence, such limit exists. Because every (ϕ_1^k, ϕ_2^k) is in $\mathcal{M}$ and is optimal, so is (ϕ_1, ϕ_2). Because of (B.1) and the maximum theorem, we know

$$\phi_i(s_{-i}) = \max \Gamma_i(\sigma_{-i}(s_{-i}; \underline{\phi}_i(\,\cdot\,), \bar{\phi}_i(\,\cdot\,))),\ \forall s_{-i} \in [0,1].$$

Therefore, (ϕ_1, ϕ_2) is an optimal mechanism such that (i) it is in $\mathcal{M}$ and (ii) it is a pair of pointwise mutual one-sided optimal delegation rules. This completes the proof. □

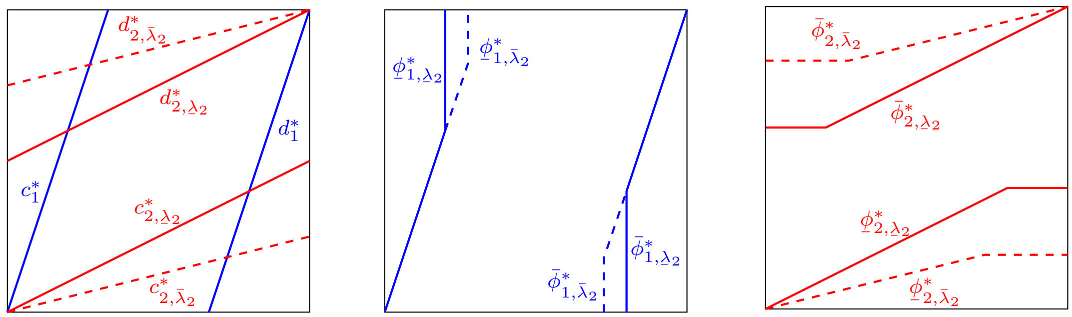
Online Appendix C Supplemental materials for Section 5.2

C.1 Comparative statics

Relative importance and optimal discretion One of the central questions in the single-agent delegation literature is how the conflict of interests between the principal and the agent affects the principal's optimal mechanism. In general, less conflict of interests leads to more discretion for the agent, e.g., Holmström (1984), Armstrong (1995), and Alonso and Matouschek (2008). In our two-agent setting, conflict of interests is measured by how important the principal thinks the agents' adaptation relative to coordination, and is represented by parameters λ_1 and λ_2. The following proposition generalizes the single-agent classical result to our two-agent setting.

Proposition C.1. *As agent i's adaptation becomes more important to the principal, i.e., λ_i increases, he will be granted more discretion, i.e., $\underline{\phi}_i^*$ shifts downwards and $\bar{\phi}_i^*$ shifts upwards. In contrast, agent $-i$ will suffer from less discretion, i.e., $\underline{\phi}_{-i}$ shifts upwards and $\bar{\phi}_{-i}$ shifts downwards.*

As the optimal mechanism is constructed by modifying the unilaterally constrained delegation rules, to understand how the optimal mechanism changes with respect to one's relative importance, it is crucial to understand how one's unilaterally



(a) $(c_i^*,\, d_i^*)$ (b) ϕ_1^* (c) ϕ_2^*

Figure C.1: Relative importance and optimal discretion: $\bar{\lambda}_2 > \underline{\lambda}_2$

constrained delegation rule changes. Lemma C.2 in the appendix provides a characterization. Panel (a) of Figure C.1 illustrates the result. As λ_2 increases from $\underline{\lambda}_2$ to $\bar{\lambda}_2$, the unilaterally coordinated delegation rule (c_2^*, d_2^*) moves in the opposite direction. The lower bound c_2^* becomes lower while the upper bound d_2^* becomes higher. From panel (a), we can derive how the optimal delegation rules ϕ_1^* and ϕ_2^* change, which are illustrated in panels (b) and (c) respectively. From panel (c), we see that agent 2 will indeed enjoy more discretion as λ_2 increases. In contrast, from panel (b), we see that agent 1 will suffer from increases in λ_2, as the delegation bounds for him becomes tighter. This is also intuitive. As agent 2 gains more discretion, he is more likely to choose his most preferred action. Agent 1 then has to carry more burden of coordination. This is done by granting agent 1 less discretion.

As a simple corollary of Proposition C.1, consider the case where agent 1 and 2's state distributions are identical. If they are equally important to the principal, i.e., $\lambda_1 = \lambda_2$, the optimal delegation rules for them will be symmetric. But if one agent is more important than the other to the principal, then she will favor the more important agent by granting more discretion at the other agent's cost of receiving less discretion.

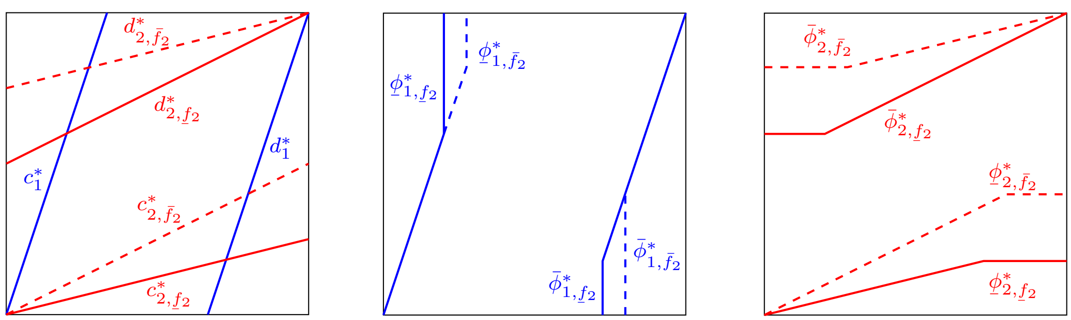
Because a change in one's relative importance has opposite effects on the two agents, if both λ_1 and λ_2 change in the same direction, the overall effects are in general ambiguous. However, the following proposition shows that the total effects of proportional changes in λ_1 and λ_2 are definite and intuitive.

Proposition C.2. *Fix $\hat{\lambda}_1, \hat{\lambda}_2 > 0$. Let $\lambda_i(\kappa) \equiv \kappa\hat{\lambda}_i$ for $i = 1, 2$. Denote by $(\phi^*_{1,\kappa},\, \phi^*_{2,\kappa})$ the optimal mechanism for importance parameters $\lambda_1(\kappa)$ and $\lambda_2(\kappa)$). As κ increases, both agents enjoy more discretion, i.e., $\underline{\phi}^*_{i,\kappa}$ shifts downwards and $\bar{\phi}^*_{i,\kappa}$ shifts upwards as κ increases for $i = 1, 2$.*

The intuition should be straightforward. When λ_1 and λ_2 increases proportionally, the relative importance between these two agents remain constant. But the importance of coordination, which can be measured by $\frac{1}{\kappa}$ in Proposition C.2, is decreasing. Consequently, the principal should grant more discretion to both agents in order to reduce their adaptation losses.

State distribution and optimal delegation rules Another aspect that affects the principal's optimal mechanism is her belief about the state distributions. For instance, if one agent's state distribution shifts to the right, how will the optimal mechanism respond? The next proposition provides the answer. It compares the optimal mechanisms when one agent's state distribution changes in the sense of the monotone likelihood ratio property (MLRP).

Proposition C.3. *When one agent's state distribution increases in the sense of the MLRP, the optimal delegation rules for both agents' shift upwards.*



(a) $(c^*_i,\, d^*_i)$ (b) ϕ^*_1 (c) ϕ^*_2

Figure C.2: State distribution and optimal discretion: $\bar{f}_2/\underline{f}_2$ is increasing

Lemma C.3 in the appendix analyzes how one's unilaterally constrained delegation rule changes if his state distribution increases in the sense of the MLRP. Panel (a) of Figure C.2 illustrates its result. As agent 2's state distribution changes from $\underline{f}_2$ to $\bar{f}_2$, both c^*_2 and d^*_2 shift upwards. Panels (b) and (c), which are derived from panel (a), depict how the optimal mechanism changes accordingly. Obviously, all the optimal boundary functions shift upwards too. Intuitively, the fact that the optimal delegation rule for agent 1 also shifts upwards is due to coordination motive.

C.2 Proofs

To prove Lemma 10, we need the following lemma regarding the implications of the log-concavity of density function. When f_i is log-concave, it is well-known that the functions F_i, $1-F_i$, $s_i \mapsto \int_0^{s_i} F_i(s_i')\mathrm{d}s_i'$ and $s_i \mapsto \int_{s_i}^1 (1-F_i(s_i'))\mathrm{d}s_i'$ inherit log-concavity.[19] The following lemma lists some related properties which will be used in the proof of Lemma 10. Part (iii) strengthens the log-concavity of $\int_0^{s_i} F_i(s_i')\mathrm{d}s_i'$ and $\int_{s_i}^1 (1-F_i(s_i'))\mathrm{d}s_i'$ to be strict.

Lemma C.1. *Assume f_i is strictly positive, log-concave, and continuous over $(0,1)$.*

(i) $\frac{f_i}{F_i}$ is decreasing and $\lim_{s_i \downarrow 0} \frac{f_i(s_i)}{F_i(s_i)} = +\infty$.

(ii) $\frac{f_i}{1-F_i}$ is increasing and $\lim_{s_i \uparrow 1} \frac{f_i(s_i)}{1-F_i(s_i)} = +\infty$.

(iii) Both $\int_0^{s_i} F_i(s_i')\mathrm{d}s_i'$ and $\int_{s_i}^1 (1-F_i(s_i'))\mathrm{d}s_i'$ are strictly log-concave. Therefore, $\frac{F_i(s_i)}{\int_0^{s_i} F_i(s_i')\mathrm{d}s_i'}$ is strictly decreasing and $\frac{1-F_i(s_i)}{\int_{s_i}^1 (1-F_i(s_i'))\mathrm{d}s_i'}$ is strictly increasing.

Proof. For parts (i) and (ii), monotonicity directly comes from log-concavity of F_i and $1-F_i$. We now show $\lim_{s_i \uparrow 1} \frac{f_i(s_i)}{1-F_i(s_i)} = \infty$. The other limit is similar. For any $s_i > \frac{1}{2}$, we have

$$\int_{\frac{1}{2}}^{s_i} \frac{f_i(s_i')}{1-F_i(s_i')}\mathrm{d}s_i' = \log(1-F(\frac{1}{2})) - \log(1-F(s_i)).$$

Hence,

$$\lim_{s_i \uparrow 1} \int_{\frac{1}{2}}^{s_i} \frac{f_i(s_i')}{1-F_i(s_i')}\mathrm{d}s_i' = +\infty.$$

Because $\frac{f_i}{1-F_i}$ is increasing, we know $\lim_{s_i \uparrow 1} \frac{f_i(s_i)}{1-F_i(s_i)} = +\infty$.

For part (iii), we show that $\int_{s_i}^1 (1-F(s_i'))\mathrm{d}s_i'$ is strict log-concave. Consider any $s_i \in (0,1)$. By part (ii), we know there exists $s_i'' \in (s_i, 1)$ such that

$$\frac{f_i(s_i)}{1-F_i(s_i)} \leq \frac{f_i(s_i')}{1-F_i(s_i')}, \ \forall s_i' \in (s_i, 1),$$

with strictly inequality when $s_i' \in (s_i'', 1)$. This implies

$$\frac{f_i(s_i)}{1-F_i(s_i)} \int_{s_i}^1 (1-F_i(s_i'))\mathrm{d}s_i' < \int_{s_i}^1 \frac{f_i(s_i')}{1-F_i(s_i')}(1-F_i(s_i'))\mathrm{d}s_i' = 1-F_i(s_i),$$

[19] See, for example, Proposition 1 in An (1998), Theorems 1 and 3 in Bagnoli and Bergstrom (2005).

which in turn implies

$$\left[\log\int_{s_i}^1 (1-F_i(s_i'))\mathrm{d}s_i'\right]'' = \frac{f_i(s_i)\int_{s_i}^1(1-F_i(s_i'))\mathrm{d}s_i' - (1-F_i(s_i))^2}{\left(\int_{s_i}^1(1-F_i(s_i'))\mathrm{d}s_i'\right)^2} < 0.$$

Therefore, $\int_{s_i}^1(1-F_i(s_i'))\mathrm{d}s_i'$ is strictly log-concave. □

Proof of Lemma 10. Let

$$\pi_i(c,d,s_{-i}) \equiv \int_0^1 \left[-\lambda_i(\sigma_i(s_i;c,d)-s_i)^2 - (\sigma_i(s_i;c,d)-s_{-i})^2\right]\mathrm{d}F_i(s_i).$$

Step 1: The unique solution $(c_i^*(s_{-i}),\ d_i^*(s_{-i}))$ to $\max_{0\le c\le d\le 1}\pi_i(c,d,s_{-i})$ is determined by the following equations:[20]

$$c_i^*(s_{-i}) = s_{-i} - \lambda_i\frac{\int_0^{c_i^*(s_{-i})} F_i(s_i)\mathrm{d}s_i}{F_i(c_i^*(s_{-i}))}, \tag{C.1}$$

$$d_i^*(s_{-i}) = s_{-i} + \lambda_i\frac{\int_{d_i^*(s_{-i})}^1 (1-F_i(s_i))\mathrm{d}s_i}{1-F_i(d_i^*(s_{-i}))}. \tag{C.2}$$

It is easy to see that $\pi_i(c,d,s_{-i}) < \pi_i(c,s_{-i},s_{-i})$ if $0\le c\le d<s_{-i}$ and $\pi_i(c,d,s_{-i}) < \pi_i(s_{-i},d,s_{-i})$ if $s_{-i}<c\le d\le 1$. Therefore, any optimal solution must satisfy $c\le s_{-i}\le d$. Note that this observation immediately implies that $c_i^*(0)=0$ and $d_i^*(1)=1$.

Assume $s_{-i}>0$. It is easy to calculate that

$$\frac{\partial\pi_i(c,d,s_{-i})}{\partial c} = -2\int_0^c \lambda_i F_i(s_i)\mathrm{d}s_i - 2F_i(c)(c-s_{-i}),$$
$$\frac{\partial^2\pi_i(c,d,s_{-i})}{\partial c^2} = 2F_i(c)\left[\frac{f_i(c)}{F_i(c)}(s_{-i}-c)-(\lambda_i+1)\right].$$

Let $g(c)\equiv \frac{f_i(c)}{F_i(c)}(s_{-i}-c)-(\lambda_i+1)$. Because F_i is log-concave, $\frac{f_i}{F_i}$ is decreasing. Thus, g is strictly decreasing over $(0,s_{-i}]$. Because $\lim_{c\downarrow 0}\frac{f_i(c)}{F_i(c)}=+\infty$, we know $\lim_{c\downarrow 0} g(c)=+\infty$. Moreover, because $g(s_{-i})<0$, we know there exists $c'\in(0,s_{-i})$ such that g is positive over $(0,c')$ and negative over (c',s_{-i}). This in turn implies that $\frac{\partial\pi_i(c,d,s_{-i})}{\partial c}$ is strictly increasing over $(0,c')$ and strictly decreasing over (c',s_{-i}). Because $\lim_{c\downarrow 0}\frac{\partial\pi_i(c,d,s_{-i})}{\partial c}=0$ and $\frac{\partial\pi_i(s_{-i},d,s_{-i})}{\partial c}<0$, we know there exists $c^*\in(c',s_{-i})$ such that $\frac{\partial\pi_i(c,d,s_{-i})}{\partial c}$ is positive over $(0,c^*)$ and negative over (c^*,s_{-i}). Clearly, this

[20] Define $\frac{\int_0^0 F_i(s_i)\mathrm{d}s_i}{F_i(0)} = \frac{\int_1^1(1-F_i(s_i))\mathrm{d}s_i}{1-F_i(1)} = 0$ by continuity.

c^* is the unique optimal one over $[0, s_{-i}]$. It is determined by $\frac{\partial \pi_i(c^*, d, s_{-i})}{\partial c} = 0$, or equivalently by equation (C.1).

Using the fact that $1 - F_i$ is log-concave, we can similarly show that, for $s_{-i} < 1$, the unique optimal $d^* \in [s_{-i}, 1]$ is determined by $\frac{\partial \pi_i(c, d^*, s_{-i})}{\partial d} = 0$, or equivalently by equation (C.2).

Step 2: $c_i^* : [0,1] \to [0,1]$ strictly increases from 0 to $c_i^*(1) < 1$ and $d_i^* : [0,1] \to [0,1]$ strictly increases from $d_1^*(0) > 0$ to 1.

From (C.1), we can easily see that $c_i^*(1) < 1$. Moreover, We can rewrite (C.1) as

$$c_i^*(s_{-i}) + \lambda_i \frac{\int_0^{c_i^*(s_{-i})} F_i(s_i) \mathrm{d}s_i}{F_i(c_i^*(s_i))} = s_i.$$

By part (iii) of Lemma C.1, we know $c \mapsto c + \lambda_i \frac{\int_0^c (1 - F_i(s_i)) \mathrm{d}s_i}{F_i(c)}$ is strictly increasing. This immediately implies that c_i^* is strictly increasing. The proof for d_i^* is similar.

Step 3: $c_i^*(0) < d_i^*(0)$, $c_i^*(1) < d_i^*(1)$ and $c_i^*(s_{-i}) < s_{-i} < d_i^*(s_{-i})$ for all $s_{-i} \in (0,1)$. Therefore, Assumption U is satisfied.

By the first two steps, we know $c_i^*(0) = 0 < d_i^*(0)$ and $c_i^*(1) < 1 = d_i^*(1)$. Moreover, Step 2 implies $c_i^*(s_{-i}), d_i^*(s_{-i}) \in (0,1)$ for $s_{-i} \in (0,1)$. Therefore, (C.1) and (C.2) immediately imply $c_i^*(s_{-i}) < s_{-i} < d_i^*(s_{-i})$ for $s_i \in (0,1)$.

Step 4: The unique intersection of c_1^* and c_2^* is $(\underline{L}_1, \underline{L}_2) = (0,0)$. The unique intersection of d_1^* and d_2^* is $(\bar{H}_1, \bar{H}_2) = (1,1)$.

Clearly, $(0,0)$ is an intersection of c_1^* and c_2^* as $c_1^*(0) = c_2^*(0) = 0$. For any $s_1 > 0$, Steps 2 and 3 imply $c_1^*(c_2^*(s_1)) < c_1^*(s_1) < s_1$. Hence, $(0,0)$ is the unique intersection of c_1^* and c_2^*. The proof for d_1^* and d_2^* is similar.

Step 5: c_i^* and d_{-i}^* has a unique intersection. Therefore, Assumption R is also satisfied.

Consider c_2^* and d_1^* as an example. By (C.1) and (C.2), we have

$$\begin{aligned} d_1^*(c_2^*(s_1)) &= c_2^*(s_1) + \lambda_2 \frac{\int_{d_1^*(c_2^*(s_1))}^1 (1 - F_1(s_1')) \mathrm{d}s_1'}{1 - F_1(d_1^*(c_2^*(s_1)))} \\ &= s_1 - \lambda_1 \frac{\int_0^{c_2^*(s_1)} F_2(s_2') \mathrm{d}s_2'}{F_2(c_2^*(s_1))} + \lambda_2 \frac{\int_{d_1^*(c_2^*(s_1))}^1 (1 - F_1(s_1')) \mathrm{d}s_1'}{1 - F_1(d_1^*(c_2^*(s_1)))}, \ \forall s_1. \end{aligned}$$

Equivalently, we can write

$$d_1^*(c_2^*(s_1)) - s_1 = \lambda_2 \frac{\int_{d_1^*(c_2^*(s_1))}^1 (1 - F_1(s_1'))\mathrm{d}s_1'}{1 - F_1(d_1^*(c_2^*(s_1)))} - \lambda_1 \frac{\int_0^{c_2^*(s_1)} F_2(s_2')\mathrm{d}s_2'}{F_2(c_2^*(s_1))}. \tag{C.3}$$

Because both d_1^* and c_2^* are strictly increasing, so is $d_1^*(c_2^*(\cdot))$. Applying part (iii) of Lemma C.1, we know that the right hand side of (C.3) is strictly decreasing. Hence the left hand side is strictly decreasing too. Because $d_1^*(c_2^*(0)) = d_1^*(0) > 0$ and $d_1(c_2^*(1)) < d_1^*(1) = 1$, we know there exists a unique $\bar{L}_1 \in (0,1)$ such that $d_1^*(c_2^*(\bar{L}_1)) = \bar{L}_1$. Let $\underline{H}_2 \equiv c_2^*(\bar{x}_1)$. Then, d_1^* and c_2^* has a unique intersection at $(\bar{L}_1, \underline{H}_2)$. $\square$

Proposition C.1 is a direct implication of Lemma C.2 below. Denote by $(c_{i,\lambda_i}^*, d_{i,\lambda_i}^*)$ the unilaterally constrained delegation rule for agent i when the relative importance of his adaptation is λ_i.

Lemma C.2. *For any $s_{-i} \in (0,1)$, $c_{i,\lambda_i}^*(s_{-i})$ is strictly decreasing while $d_{i,\lambda_i}^*(s_{-i})$ is strictly increasing in λ_i. Moreover, $\lim_{\lambda_i \to \infty} c_{i,\lambda_i}^* \equiv 0$, $\lim_{\lambda_i \to \infty} d_{i,\lambda_i}^* \equiv 1$, and*

$$c_{i,0}^*(s_{-i}) = d_{i,0}^*(s_{-i}) = s_{-i}, \ \forall s_{-i}.$$

Proof of Lemma C.2. Assume $\bar{\lambda}_i > \underline{\lambda}_i$. Pick any $s_{-i} \in (0,1)$. We only show $c_{i,\bar{\lambda}_i}^*(s_{-i}) < c_{i,\underline{\lambda}_i}^*(s_{-i})$. For notational simplicity, let $\underline{c} = c_{i,\underline{\lambda}_i}^*(s_{-i})$ and $\bar{c} = c_{i,\bar{\lambda}_i}^*(s_{-i})$. By (C.1), we have

$$\underline{c} + \underline{\lambda}_i \frac{\int_0^{\underline{c}} F_i(s_i)\mathrm{d}s_i}{F_i(\underline{c})} = \bar{c} + \bar{\lambda}_i \frac{\int_0^{\bar{c}} F_i(s_i)\mathrm{d}s_i}{F_i(\bar{c})} > \bar{c} + \underline{\lambda}_i \frac{\int_0^{\bar{c}} F_i(s_i)\mathrm{d}s_i}{F_i(\bar{c})}.$$

Because $c \mapsto c + \underline{\lambda}_i \frac{\int_0^c F_i(s_i)\mathrm{d}s_i}{F_i(c)}$ is strictly increasing, we know $\underline{c} > \bar{c}$. $\square$

Proof of Proposition C.2. Consider $0 < \underline{\kappa} < \bar{\kappa} < \infty$. We show $\bar{\phi}_{1,\bar{\kappa}}^* \geq \bar{\phi}_{1,\underline{\kappa}}^*$ and $\underline{\phi}_{2,\bar{\kappa}}^* \leq \underline{\phi}_{2,\underline{\kappa}}^*$. It is most easily understood by looking at Figure C.3. Let $(\bar{L}_{1,\kappa}, \underline{H}_{2,\kappa})$ be the intersection of $d_{1,\kappa}^*$ and $c_{2,\kappa}^*$ for $\kappa \in \{\underline{\kappa}, \bar{\kappa}\}$. By Lemma C.2, we know $d_{1,\bar{\kappa}}^* \geq d_{1,\underline{\kappa}}^*$ and $c_{2,\bar{\kappa}}^* \leq c_{2,\underline{\kappa}}^*$. Hence $(\bar{L}_{1,\bar{\kappa}}, \underline{H}_{2,\bar{\kappa}})$ can only appear in one of the regions i, ii, iii in Figure C.3.

We claim that, in fact, $(\bar{L}_{1,\bar{\kappa}}, \underline{H}_{2,\bar{\kappa}})$ can only be in region iii. To see this, consider the equations that determine the intersections, i.e., (C.3) in the proof of Lemma 10 and the following analysis. We have

$$\begin{aligned} 0 &= \underline{\kappa}\hat{\lambda}_2 \frac{\int_{\bar{L}_{1,\underline{\kappa}}}^1 (1 - F_1(s_1))\mathrm{d}s_1}{1 - F_1(\bar{L}_{1,\underline{\kappa}})} - \underline{\kappa}\hat{\lambda}_1 \frac{\int_0^{\underline{H}_{2,\underline{\kappa}}} F_2(s_2)\mathrm{d}s_2}{F_2(\underline{H}_{2,\underline{\kappa}})} \\ &= \bar{\kappa}\hat{\lambda}_2 \frac{\int_{\bar{L}_{1,\bar{\kappa}}}^1 (1 - F_1(s_1))\mathrm{d}s_1}{1 - F_1(\bar{L}_{1,\bar{\kappa}})} - \bar{\kappa}\hat{\lambda}_1 \frac{\int_0^{\underline{H}_{2,\bar{\kappa}}} F_2(s_2)\mathrm{d}s_2}{F_2(\underline{H}_{2,\bar{\kappa}})}. \end{aligned}$$

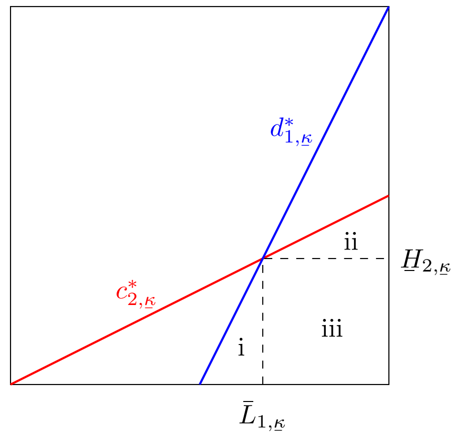


Figure C.3: Graph for the proof of Proposition C.2

Because $x \mapsto \frac{\int_x^1 (1-F_1(s_1))\mathrm{d}s_1}{1-F_1(x)}$ is strictly decreasing and $x \mapsto \frac{\int_0^x F_2(s_2)\mathrm{d}s_2}{F_2(x)}$ is strictly increasing by Lemma C.1, it is easy to see from the above equation that we can have neither $\bar{L}_{1,\bar{\kappa}} \leq \bar{L}_{1,\underline{\kappa}}$ and $\underline{H}_{2,\bar{\kappa}} < \underline{H}_{2,\underline{\kappa}}$, nor $\bar{L}_{1,\bar{\kappa}} > \bar{L}_{1,\underline{\kappa}}$ and $\underline{H}_{2,\bar{\kappa}} \geq \underline{H}_{2,\underline{\kappa}}$. In other words, $(\bar{L}_{1,\bar{\kappa}},\ \underline{H}_{2,\bar{\kappa}})$ can be in neither region i nor region ii.

Therefore, $(\bar{L}_{1,\bar{\kappa}},\ \underline{H}_{2,\bar{\kappa}})$ is in region iii. Equivalently, $\bar{L}_{1,\bar{\kappa}} \geq \bar{L}_{1,\underline{\kappa}}$ and $\underline{H}_{2,\bar{\kappa}} \leq \underline{H}_{2,\underline{\kappa}}$. For any $s_2 \in [0,1)$, we have

$$\bar{\phi}^*_{1,\bar{\kappa}}(s_1) = \max\{d^*_{1,\bar{\kappa}}(s_1),\ \bar{L}_{1,\bar{\kappa}}\} \geq \max\{d^*_{1,\underline{\kappa}}(s_1),\ \bar{L}_{1,\underline{\kappa}}\} = \bar{\phi}^*_{1,\underline{\kappa}}(s_1).$$

Similarly, for any $s_1 \in (0,1]$, we have

$$\underline{\phi}^*_{2,\bar{\kappa}}(s_2) = \min\{c^*_{2,\bar{\kappa}}(s_2),\ \underline{H}_{2,\bar{\kappa}}\} \leq \min\{c^*_{2,\underline{\kappa}}(s_2),\ \underline{H}_{2,\underline{\kappa}}\} = \underline{\phi}^*_{2,\underline{\kappa}}(s_2). \qquad \square$$

Proposition C.3 is a direct implication of Lemma C.3 below. Denote by $(c^*_{i,f_i},\ d^*_{i,f_i})$ i's unilaterally coordinated delegation rule when his state distribution is f_i.

Lemma C.3. *Suppose $0 < \lambda_i < \infty$. Consider two densities $\underline{f}_i$ and $\bar{f}_i$ of agent i's state distribution. If the likelihood ratio $\bar{f}_i/\underline{f}_i$ is (strictly) increasing, then $c^*_{i,\bar{f}_i}(s_{-i}) \geq (>)\, c^*_{i,\underline{f}_i}(s_{-i})$ and $d^*_{i,\bar{f}_i}(s_{-i}) \geq (>)\, d^*_{i,\underline{f}_i}(s_{-i})$ for all $s_{-i} \in (0,1)$.*

Proof of Lemma C.3. Let $\bar{F}_i$ and $\underline{F}_i$ be the c.d.f.'s of $\bar{f}_i$ and $\underline{f}_i$ respectively. Because $\bar{f}_i$ and $\underline{f}_i$ satisfy the (strict) MLRP, we know that, for all $c, d \in (0,1)$,[21]

$$\frac{\int_0^c \bar{F}_i(s_i)\mathrm{d}s_i}{\bar{F}_i(c)} \leq (<) \frac{\int_0^c \underline{F}_i(s_i)\mathrm{d}s_i}{\underline{F}_i(c)} \text{ and } \frac{\int_d^1 (1-\bar{F}_i(s_i))\mathrm{d}s_i}{1-\bar{F}_i(s_i)} \geq (>) \frac{\int_d^1 (1-\underline{F}_i(s_i))\mathrm{d}s_i}{1-\underline{F}_i(s_i)}.$$

[21]See, for example, Theorem 1.C.1 in Shaked and Shanthikumar (2007).

Consider $s_{-i} \in (0,1)$. Let $\underline{c} = c^*_{i,\underline{f}_i}(s_{-i})$ and $\bar{c} = c^*_{i,\bar{f}_i}(s_{-i})$. By (C.1), we have

$$\underline{c} + \lambda_i \frac{\int_0^{\underline{c}} \underline{F}_i(s_i)\mathrm{d}s_i}{\underline{F}_i(\underline{c})} = \bar{c} + \lambda_i \frac{\int_0^{\bar{c}} \bar{F}_i(s_i)\mathrm{d}s_i}{\bar{F}_i(\bar{c})} \leq (<)\, \bar{c} + \lambda_i \frac{\int_0^{\bar{c}} \underline{F}_i(s_i)\mathrm{d}s_i}{\underline{F}_i(\bar{c})}.$$

Again, because $c \mapsto c + \lambda_i \frac{\int_0^{c} \underline{F}_i(s_i)\mathrm{d}s_i}{\underline{F}_i(c)}$ is strictly increasing, we know $\underline{c} \leq (<)\, \bar{c}$. $\square$