SUPPLEMENT TO

"Dynamic Pricing in the Presence of Individual Learning"

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1 Equilibrium Analysis in the Presence of Common Uncertainty

In the paper, the buyer's valuations are purely idiosyncratic. Common uncertainty can be introduced in the following way. The arrival of lump-sum payoffs depends on both an intrinsic characteristic of the product (*common uncertainty*) and the quality of the match between the product and that buyer (*idiosyncratic uncertainty*). The product characteristic is either *high* ($\lambda = \lambda_H$) or low ($\lambda = \lambda_L = 0$), and the match between buyer i and the risky product is either *relevant* ($\kappa_i = 1$) or *irrelevant* ($\kappa_i = 0$). The arrival of random lump-sum payoffs ξ_l is independent across buyers and modeled as a Poisson process with intensity $\lambda\kappa_i$. Therefore, a buyer i is able to receive random lump-sum payoffs if and only if both the product characteristic is high and the individual match quality is relevant. Before the game starts, nature chooses randomly and independently the product characteristic and the individual match quality for each buyer. The common priors are such that: $q_0 = \Pr(\lambda = \lambda_H)$, and for each buyer i, $\rho_0 = \Pr(\kappa_i = 1)$. The product characteristic and the match qualities are initially unobservable to all players (seller and buyers), but the parameters λ_H, ξ_f, ξ_l, ρ_0 and q_0 are common knowledge.

Given a public history h_t, posterior beliefs are defined as:

$$q_t \triangleq \Pr[\lambda_H \mid h_t] \quad \text{and} \quad \rho_{it} \triangleq \Pr[\kappa_i = 1 \mid \lambda_H, h_t]$$

such that the posterior belief of receiving lump-sum payoffs is given by

$$\Pr[\lambda\kappa_i = \lambda_H \mid h_t] = \rho_{it} q_t.$$

Given a pair of priors (ρ_0, q_0), the posteriors $(\rho_{1t}, \cdots, \rho_{nt}, q_t)$ evolve according to Bayes' rule. While the evolution of ρ_{it} is given by

$$\dot{\rho}_{it} = -\lambda_H a_{it} \rho_{it}(1 - \rho_{it}), \tag{1}$$

there is also updating about q_t. If no buyer has received a lump-sum payoff, then with an expected arrival rate $\lambda_H q_t \sum_{i=1}^n a_{it}\rho_{it}$, some buyer receives a lump-sum payoff and q_t jumps to 1. Otherwise, q_t obeys the following differential equation at those times when a_{it} is right continuous for $\forall i$:

$$\dot{q}_t = -\lambda_H q_t (1-q_t) \sum_{i=1}^n a_{it}\rho_{it}. \tag{2}$$

The posterior belief q can be expressed as a function of ρ_i's. When no buyer has received a lump-sum payoff for a length of time t, let $x_{it} \triangleq \rho_0 e^{-\lambda_H \int_0^t a_{i\tau} d\tau} + 1 - \rho_0$ denote the probability of the event that unknown buyer i has not received lump-sum payoffs for a length of time t conditional on λ_H. By Bayes' rule

$$q_t = \frac{q_0 \prod_{i=1}^n x_{it}}{q_0 \prod_{i=1}^n x_{it} + 1 - q_0}. \tag{3}$$

From equation (1),

$$\rho_{it} = \frac{\rho_0 e^{-\lambda_H \int_0^t a_{i\tau} d\tau}}{x_{it}} \Longrightarrow 1 - \rho_{it} = \frac{1-\rho_0}{x_{it}}. \tag{4}$$

Substituting (4) into (3) yields:

$$q_t = \frac{q_0(1-\rho_0)^n}{q_0(1-\rho_0)^n + (1-q_0)\prod_{i=1}^n (1-\rho_{it})}. \tag{5}$$

Notice that equation (5) also holds when at least one buyer has received lump-sum payoffs. In that situation, at least one of the ρ_{it}'s is one and q_t is also one. After a long time during which no lump-sum payoffs occur, the posteriors ρ_{it} would converge to zero while q_t would not. This reflects the fact that ρ_{it} is a conditional probability and q_t is bounded below by $q_0(1-\rho_0)^n$.

A nice property about equation (5) is that it only depends on ρ_{it}'s and does not explicitly depend on previous purchasing decisions or time t. Differential equations (1) and (2) imply: given a particular history of purchasing decisions, both ρ_{it} and q_t can be written as a function of time. In the critical history when no lump-sum payoffs have been received, ρ_{it} is sufficient to encode time t and the relevant information about previous purchasing decisions, which are needed to update q_t. Therefore, we can express q_t as a function of $\boldsymbol{\rho}_t \triangleq (\rho_{1t}, \cdots, \rho_{nt})$ for a given pair of priors (ρ_0, q_0).

The above discussions imply that we can still use $\boldsymbol{\rho}_t \triangleq (\rho_{1t}, \cdots, \rho_{nt})$ as the state variables to define the Markov perfect equilibrium. The equilibrium definition is the same as the one presented in the paper, and hence is omitted.

1.1 Socially Efficient Allocation in the Good News Case

In this appendix, we will focus on the good news case. The results for the bad-news model extend to the $q_0 < 1$-case as well, as the previous working paper version has shown. We present the main results without showing the proofs, which are similar to those in Appendix B, and can be found in the previous working paper.

Since the arrival of one lump-sum payoff immediately resolves common uncertainty, there are only two situations to consider: a *social learning phase*, where the common uncertainty has not been resolved, and an *individual learning phase*, where the common uncertainty has been resolved. Obviously, in the individual learning phase, when k buyers have received lump-sum payoffs, it is socially efficient for the remaining $n-k$ unknown buyers to stop purchasing the risky product once the posterior belief ρ reaches

$$\rho_I^e = \rho^e = \frac{rs}{(r+\lambda_H)g - \lambda_H s}$$

and still no lump-sum payoff has been received. The value function for a buyer with posterior belief ρ is

$$W(\rho) = \max\left\{s, g\rho + \frac{\lambda_H s}{r+\lambda_H}(\frac{rs}{(r+\lambda_H)(g-s)})^{r/\lambda_H}(1-\rho)(\frac{1-\rho}{\rho})^{r/\lambda_H}\right\}. \tag{6}$$

In the social learning phase, the socially efficient allocation solves the symmetric cooperative problem:

$$\Omega_S(\rho) = \sup_{\alpha(\cdot)\in\{0,1\}} \mathbb{E}\left\{\int_{t=0}^{h} re^{-rt} n[\alpha(\rho_t)\rho_t q(\rho_t)g + (1-\alpha(\rho_t))s]dt + e^{-rh}\Omega(\rho_h \mid \alpha)\right\}$$

where

$$\mathbb{E}\Omega(\rho_h \mid \alpha) = q\sum_{k=1}^{n}\binom{n}{k}\rho^k\left(1-e^{-\lambda_H\int_0^h \alpha_t dt}\right)^k\left(\rho e^{-\lambda_H\int_0^h \alpha_t dt} + 1 - \rho\right)^{n-k}\Omega_k(\rho_h)$$
$$+\left[q\left(\rho e^{-\lambda_H\int_0^h \alpha_t dt} + 1-\rho\right)^n + 1 - q\right]\Omega_S(\rho_h)$$

and

$$\rho_h = \frac{\rho e^{-\lambda_H\int_0^h \alpha_t dt}}{\rho e^{-\lambda_H\int_0^h \alpha_t dt} + 1 - \rho}.$$

In the continuous time framework, the probability that more than two buyers receive lump-sum payoffs at the same time is zero. The Hamilton-Jacobi-Bellman equation (HJB equation hereafter) for the above problem hence is simplified as:

$$r\Omega_S(\rho) = \max\left\{rns, rn\rho q(\rho)g + n\rho q(\rho)\lambda_H(\Omega_1(\rho) - \Omega_S(\rho)) - \lambda_H\rho(1-\rho)\Omega_S'(\rho)\right\}, \tag{7}$$

where $\Omega_1(\rho) = g + (n-1)W(\rho)$ is the social surplus when one buyer receives a lump-sum payoff.

The first part of the maximand corresponds to using the safe product, the second to the risky product. The social consequences of using the risky product can be decomposed into three elements: i) the (normalized) expected payoff rate $rn\rho q(\rho)g$, ii) the jump of the value function to $\Omega_1(\cdot)$ if one buyer receives a lump-sum payoff, which occurs at rate $n\lambda_H$ with probability $pq(\rho)$, and iii) the effect of Bayesian updating on the value function when no lump-sum payoff is received. When no lump-sum payoff is received, both ρ and q are updated. The updating of q is implicitly incorporated as a function of ρ.

The optimal cutoff ρ_S^e is pinned down by solving the following differential equation:

$$r\Omega_S(\rho) = rn\rho q(\rho)g + n\rho q(\rho)\lambda_H(\Omega_1(\rho) - \Omega_S(\rho)) - \lambda_H\rho(1-\rho)\Omega_S'(\rho), \tag{8}$$

with boundary conditions:

$$\Omega_S(\rho_S^e) = ns \quad \text{(value matching condition)} \quad \text{and} \quad \Omega_S'(\rho_S^e) = 0 \quad \text{(smooth pasting condition)}.$$

Substitute the two boundary conditions into differential equation (8) and we immediately show that the cutoff ρ_S^e should satisfy

$$rn\rho q(\rho)g + n\rho q(\rho)\lambda_H\Omega_1(\rho) = (r + n\rho q(\rho)\lambda_H)ns. \tag{9}$$

Equation (9) implies a unique solution ρ_S^e for a given pair of priors (ρ_0, q_0). The socially efficient allocation in the social learning phase can be characterized as follows:

Proposition 1 *(Characterize socially efficient allocation) For any posteriors (ρ, q), it is socially efficient to purchase the risky product in the social learning phase if and only if*

$$\rho q > \frac{rs}{(r+\lambda_H)g + (n-1)\lambda_H W(\rho) - n\lambda_H s}.$$

When the common uncertainty is resolved, it is always socially efficient for the unknown buyers to continue experimentation until the posterior reaches ρ_I^e.

Given the priors, the unique pair of efficient cutoffs $(\rho_S^e(\rho_0, q_0), q_S^e(\rho_0, q_0))$ is determined by equations

$$q_S^e = \frac{(1-\rho_0)^n q_0}{(1-\rho_0)^n q_0 + (1-\rho_S^e)^n(1-q_0)} \tag{10}$$

and

$$q_S^e = \frac{rs}{\rho_S^e[(r+\lambda_H)g + (n-1)\lambda_H W(\rho_S^e) - n\lambda_H s]}, \tag{11}$$

where $W(\cdot)$ is given by equation (6). Unlike the individual learning phase, the cutoff ρ_S^e does depend on the priors (ρ_0, q_0). We formulate the problem so that ρ is the unique state variable in order to avoid solving partial differential equations. But the actual optimal stopping decision depends not only on belief ρ but also on q. For a fixed ρ_0, a higher q_0 means that the society can afford to experiment more and thus the efficient cutoff ρ_S^e should be lower. For a fixed pair of priors (ρ_0, q_0), a two-dimensional optimal stopping problem is transformed into a one-dimensional one by expressing q as a function of ρ. As a result, we are able to apply traditional value matching and smooth pasting conditions to solve our optimal stopping problems.

1.2 Characterizing Equilibrium for $n = 2$

As in the previous analysis, the equilibrium purchasing behavior can be characterized by two cutoffs $\rho_S^\star$ and $\rho_I^\star$. If no buyer has received lump-sum payoffs, the price is falling over time to keep both unknown buyers experimenting until posterior ρ reaches $\rho_S^\star$. After that, both buyers purchase the safe product. If one buyer has received a lump-sum payoff, the monopolist stops selling to the unknown buyer and only serves the known buyer when posterior belief about the unknown buyer is below $\rho_I^\star$.

Obviously, $\rho_I^\star$ is the same as the one characterized in the main text (i.e., $\rho_I^\star \triangleq \frac{r(g+s)}{2rg+\lambda_H(g-s)}$). For $\rho_S^\star$, we can apply similar techniques developed in the paper to show the following proposition:

Proposition 2 *Fix the monopolist's strategy such that $\rho_S^\star$ is the equilibrium cutoff in the social learning phase. If, in equilibrium, both unknown buyers continue to experiment at posterior belief $\rho > \rho_S^\star$, the value $U_S(\rho)$ satisfies differential equations*

$$0 = 2(r+\lambda_H\rho q)(U_S(\rho)-s)+\lambda_H\rho(1-\rho)U_S'(\rho)+(r+\lambda_H\rho)g(1-\rho)q(\frac{(1-\rho)\rho_I^\star}{\rho(1-\rho_I^\star)})^{r/\lambda_H}-\lambda_H g\rho(1-\rho)q$$
$$-\left[\frac{r+\lambda_H\rho_S^\star}{1-\rho_S^\star}(\frac{\rho_I^\star}{1-\rho_I^\star})^{r/\lambda_H}-\lambda_H(\frac{\rho_S^\star}{1-\rho_S^\star})^{1+r/\lambda_H}\right]g(1-\rho)^2q(\frac{1-\rho}{\rho})^{r/\lambda_H}, \quad (12)$$

if $\rho_S^\star > \rho_I^\star$.
If $\rho_S^\star \leq \rho_I^\star$, the value $U_S(\rho)$ satisfies differential equation

$$0 = 2(r+\lambda_H\rho q)(U_S(\rho)-s)+\lambda_H\rho(1-\rho)U_S'(\rho)$$
$$+\frac{r\lambda_H g}{r+\lambda_H}\frac{(1-\rho)^2q\rho_S^\star}{1-\rho_S^\star}(\frac{(1-\rho)\rho_S^\star}{\rho(1-\rho_S^\star)})^{r/\lambda_H}-\frac{rg}{r+\lambda_H}\lambda_H\rho(1-\rho)q \quad (13)$$

for $\rho \leq \rho_I^\star$; and differential equation

$$0 = 2(r+\lambda_H\rho q)(U_S(\rho)-s)+\lambda_H\rho(1-\rho)U_S'(\rho)+(r+\lambda_H\rho)g(1-\rho)q(\frac{(1-\rho)\rho_I^\star}{\rho(1-\rho_I^\star)})^{r/\lambda_H}-\lambda_H g\rho(1-\rho)q$$
$$-r\left[\frac{r+\lambda_H+\lambda_H\rho_I^\star}{(r+\lambda_H)(1-\rho_I^\star)}(\frac{\rho_I^\star}{1-\rho_I^\star})^{r/\lambda_H}-\frac{\lambda_H}{r+\lambda_H}(\frac{\rho_S^\star}{1-\rho_S^\star})^{1+r/\lambda_H}\right]g(1-\rho)^2q(\frac{1-\rho}{\rho})^{r/\lambda_H} \quad (14)$$

for $\rho > \rho_I^\star$.
Meanwhile, if the value $U_S(\rho)$ satisfies the above differential equations, then the unknown buyers will keep experimenting in equilibrium.

The main difference between the independent case and the correlated case lies in the fact that it is possible to have $\rho_S^\star > \rho_I^\star$ with common uncertainty. Based on the above Proposition 2, it is straightforward to characterize the symmetric Markov perfect equilibrium when $n = 2$.

Proposition 3 *(Characterize the symmetric Markov perfect equilibrium) In the social learning phase, the unknown buyers purchase the risky product at posterior beliefs* (ρ, q) *if and only if*

$$\rho q > \frac{rs}{rg + \lambda_H(V_I(\rho) + J_I(\rho)) - \lambda_H s}.$$

$\rho_S^\star > \rho_I^\star$ *occurs if and only if*

$$\frac{1 - q_0}{q_0(1 - \rho_0)^2} > \frac{g}{(1 - \rho_I^\star)s}. \tag{15}$$

Obviously, the occurrence of $\rho_S^\star > \rho_I^\star$ depends on the relative importance of social learning and individual learning. Given q_0, when ρ_0 goes up, the monopolist has higher incentives to keep the remaining unknown buyer experimenting. $\rho_S^\star > \rho_I^\star$ thus is more likely to occur as a result.

1.3 Equilibrium Efficiency in the Presence of Common Uncertainty

This section discusses the efficiency property of the symmetric Markov perfect equilibrium for an arbitrary number of buyers. It turns out that the symmetric Markov perfect equilibrium is socially efficient if and only if the buyers' valuations are perfectly correlated.

Proposition 4 *Consider a market with any* $n \geq 2$ *buyers. When the buyers' valuations are perfectly correlated* ($\rho = 1$)*, the unknown buyers will always receive value* s *in equilibrium and the symmetric Markov perfect equilibrium is efficient. However, the symmetric Markov perfect equilibrium is inefficient if* $\rho_0 < 1$.

In the common value case, there is no ex post heterogeneity. This has three important implications leading to the restoration of efficiency: 1) there is no future rent from one-shot deviation, and there is no need for the monopolist to provide extra subsidy to deter deviation; 2) the equilibrium value for the buyers is always the same as the outside option, and there is no continuation value effect; and 3) buyers are always homogeneous no matter how many of them have received lump-sum payoffs, and as a result, the monopolist is able to fully internalize the social surplus.

2 Equilibrium Analysis with a Continuum of Buyers

We can extend our analysis to a market consisting of a continuum of buyers. The total measure of buyers is normalized to one. We follow the good news model, and assume that both the match qualities and the arrival of good news signals are independent across buyers.

It is natural to conjecture that in equilibrium, the monopolist will first set a low price to sell to both the unknown and known buyers, and eventually will set a high price to sell only to the known buyer. If the monopolist keeps selling to the unknown buyers, then the total measure of known buyers by time t is $\rho_0(1-e^{-\lambda_H t})$. For an unknown buyer i at time t, the posterior probability that $\kappa_i = 1$ is $\rho_t = \frac{\rho_0 e^{-\lambda_H t}}{\rho_0 e^{-\lambda_H t}+1-\rho_0}$. The total measure of known buyers can hence be expressed as a function of ρ_t:

$$\rho_0(1-e^{-\lambda_H t}) = \frac{\rho_0 - \rho_t}{1-\rho_t}.$$

As in the previous analysis, we use the posterior belief ρ as the unique state variable, and let $P(\rho)$ be price charged at belief ρ when the monopolist sells to both the unknown and known buyers. Then the value functions of the monopolist, the known buyers, and the unknown buyers can be respectively written as

$$rJ(\rho) = rP(\rho) - \lambda_H \rho(1-\rho)J'(\rho);$$

$$rV(\rho) = r(g - P(\rho)) - \lambda_H \rho(1-\rho)V'(\rho);$$

and

$$rU(\rho) = r(g\rho - P(\rho)) + \lambda_H \rho(V(\rho) - U(\rho)) - \lambda_H \rho(1-\rho)U'(\rho).$$

The last two differential equations imply

$$r(V-U) = rg(1-\rho) - \lambda\rho(V-U) - \lambda_H \rho(1-\rho)(V-U)',$$

which has a general solution

$$V(\rho) - U(\rho) = g(1-\rho) - D_1(1-\rho)\left(\frac{1-\rho}{\rho}\right)^{r/\lambda_H}.$$

The coefficient D_1 in pinned down by the fact that at belief x where the monopolist stops experimentation, $V(x) - U(x) = 0$. Hence, we obtain

$$V(\rho) - U(\rho) = g(1-\rho) - g(1-\rho)\left(\frac{(1-\rho)x}{\rho(1-x)}\right)^{r/\lambda_H}.$$

The price $P(\rho)$ is set to deter "one-shot" deviations. By deviating to the safe product for h length of time, an unknown buyer with belief ρ receives value

$$\hat{U}(\rho; h) = \int_{t=0}^{h} re^{-rt} s dt + e^{-rh} U^D(\rho, \rho_h),$$

where $U^D(\rho, \rho_h)$ is the deviator's continuation value from purchasing the risky product after the deviation. Obviously, $U^D(\rho, \rho_h)$ is strictly larger than $U(\rho_h)$ since the deviator is more optimistic: $\rho > \rho_h$.

Let $\rho(t)$ and $\rho_h(t)$ be posterior beliefs after t length of time beginning from ρ and ρ_h (given that no lump-sum payoff is received during this period). Denote $Z(t) = U^D(\rho(t), \rho_h(t)) - U(\rho(t))$, and $Z(t)$ satisfies the following differential equation

$$(r + \lambda_H \rho(t))\, Z(t) - \dot{Z}(t) = r\left(\rho(t) - \rho_h(t)\right) g + \lambda_H \left(\rho(t) - \rho_h(t)\right)\left(V(\rho_h(t)) - U(\rho_h(t))\right).$$

Plugging the expression of $V(\rho) - U(\rho)$, we can solve $Z(t)$ as

$$Z(t) = \left(\rho(t) - \rho_h(t)\right) g - g\left(1 - \rho_h(t)\right)\left(\frac{(1-\rho_h(t))x}{\rho_h(t)(1-x)}\right)^{r/\lambda_H} + D_2\left(1 - \rho(t)\right)\left(\frac{1-\rho(t)}{\rho(t)}\right)^{r/\lambda_H}.$$

D_2 is chosen such that $Z(t) = 0$ when $\rho_h(t) = x$. This implies

$$Z(0) = U^D(\rho, \rho_h) - U(\rho_h) = (\rho - \rho_h) g\left[1 - \left(\frac{(1-\rho_h)x}{\rho_h(1-x)}\right)^{r/\lambda_H}\right].$$

It is optimal for the monopolist to set price $P(\rho)$ such that

$$\lim_{h\to 0} \frac{U(\rho) - \hat{U}(\rho; h)}{h} = r(U(\rho) - s) + \lim_{h\to 0} \frac{U(\rho) - U^D(\rho, \rho_h)}{h} = 0,$$

which implies that

$$r(U(\rho) - s) + \lambda_H \rho(1-\rho)U'(\rho) - \lambda_H \rho(1-\rho) g\left[1 - \left(\frac{(1-\rho)x}{\rho(1-x)}\right)^{r/\lambda_H}\right] = 0.$$

Plugging this equation into the expression of $U(\rho)$ yields: $P(\rho) = g\rho - s$. Therefore, we obtain the following proposition.

Proposition 5 *Fix any ρ_0. Denote $x^\star < \rho_0$ to be the unique solution to equation*

$$rgx^2 - \left[2rg + \lambda_H(1-\rho_0)(g-s)\right] x + rs + r\rho_0(g-s) = 0.$$

In equilibrium, the monopolist sets price $g\rho - s$ when $\rho > x^\star$, and price $g - s$ when ρ reaches $x^\star$.

It might be surprising to observe the price $P(\rho)$ is $g\rho - s$, which makes the unknown buyers myopically indifferent between purchasing and not purchasing. The basic intuition can be understood by a two period model. Suppose the monopolist sells to the unknown buyers in both periods. Then the equilibrium price in the second period is obviously $P_2 = g\rho_2 - s$, where ρ_2 is the posterior belief of the unknown buyers in the second period. If an unknown buyer purchases the risky product in the first period, the expected payoff in the second period hence is

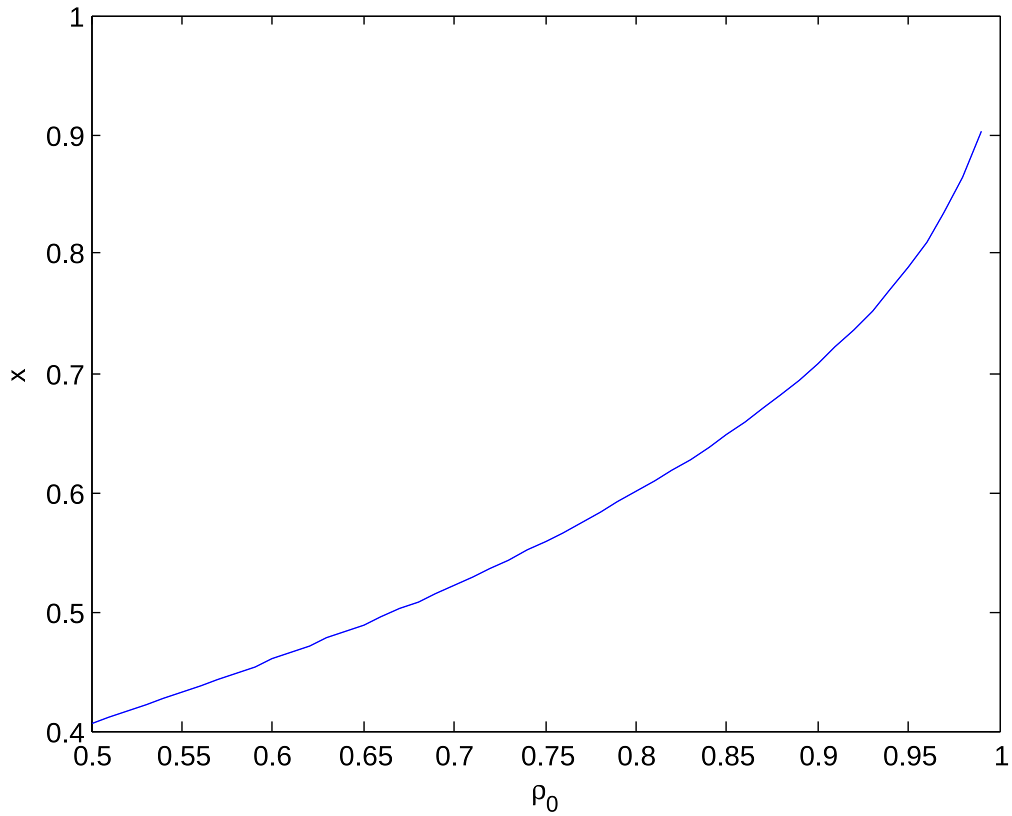

Figure 1: $x^{\star}$ as a Function of ρ_0.

$$\rho_1 y(g - P_2) + (1 - \rho_1 y)s = s + \rho_1 y(1 - \rho_2)g,$$

where ρ_1 is the prior that the buyer has match quality $\kappa = 1$, and y is the probability that the $\kappa = 1$ buyer receives a good news signal in the first period. If the buyer does not purchase, the expected payoff in the second period is just $\rho_1 g - P_2 = s + (\rho_1 - \rho_2)g$. Since $\rho_2 = \frac{\rho_1(1-y)}{1-\rho_1 y}$, it is straightforward to verify that $\rho_1 y(1 - \rho_2) = \rho_1 - \rho_2$. Therefore, the expected payoff in the second period stays the same no matter whether the unknown purchases or not in the first period. As a result, the deterrence effect and the continuation value effect cancel each other out, and the price set in the first period must be $g\rho_1 - s$. Although the unknown buyers are myopically indifferent, the equilibrium value $U(\rho)$ is strictly larger than s, because the possibility of receiving a good news signal and obtaining a higher value $V(\rho)$. This is similar to our main result: the unknown buyers receive a rent to deter deviations.

It is also worthwhile pointing out that the equilibrium cutoff $x^{\star}$ depends on the prior ρ_0. By stopping experimentation at x, the monopolist receives value $\frac{\rho_0 - x}{1-x}(g - s)$ afterwards. Therefore, a higher ρ_0 makes it more attractive to stop at a higher $x^{\star}$ as shown by Fig. 1.