Can Learning Cause Shorter Delays in Reaching Agreements?☆

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Abstract

This paper uses a continuous-time war of attrition model to investigate how learning about private payoffs affects delays in reaching agreement. At each point in time, players may receive a private Poisson signal that completely reveals the concession payoff to be high (good-news learning) or low (bad-news learning). In the good-news model, the expected delay is always non-monotonic in the learning rate: an increase in the learning rate prolongs delay in agreement if the learning rate is sufficiently low. In the bad-news model, numerical examples suggest learning prolongs delay as well.

Keywords: Learning; War of Attrition; Delay; Incomplete Information.
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1. Introduction

This paper develops a continuous-time war of attrition model with learning to investigate how learning affects delays in reaching agreement. In the model, two risk-neutral players decide when to concede. A player receives a high deterministic payoff if her opponent concedes first, while there is uncertainty about her payoff if she concedes first. As long as no player concedes, each player has a chance of privately learning whether her concession payoff is high or low (normalized to 0). For technical tractability, learning is modeled in a very stylized way: at each point in time, each player may receive a private Poisson signal that completely reveals the concession payoff. This paper focuses on two different ways of interpreting the Poisson signal. In the good-news (resp. bad-news) model, the signal reveals a high (resp. low) concession payoff, which increases (resp. decreases) the receiver's incentive to concede. For further simplicity, we also assume full symmetry between the two players; hence, we can focus on symmetric equilibrium of the game as in [3].

In the presence of learning, the war of attrition starts as a complete information game, but due to learning, incomplete information about the payoffs may develop over time. Due to the Poisson signal structure, a player can be either informed or uninformed about her concession payoff at any point of time, and one type of player is more willing to concede than the other. This enables us to fully characterize the unique symmetric equilibrium of the game.

The paper first compares the uninformed benchmark, in which there is no revelation of the concession payoff, with the full-information benchmark, in which there is immediate revelation of the concession payoff. It is shown that the expected delay is always shorter in the full-information benchmark. Since these two benchmarks correspond to the special cases in which the learning rate is 0 and infinity, respectively, one may conjecture that an increase in the learning rate always leads to shorter delays in reaching agreements. This conjecture is shown to be incorrect: the expected delay can be non-monotonic in the learning rate, and learning can cause longer delays in reaching agreements.

In the good-news model, we show that, at any point in time, equilibrium play falls into one of three possible cases. In the first case, an uninformed player randomizes between conceding and staying, while an informed player strictly prefers conceding immediately. In the second case, an uninformed player strictly prefers staying, while an informed player randomizes between conceding and staying. In the third case, an uninformed player strictly prefers staying, while an informed player strictly prefers conceding immediately. The equilibrium characterization depends on the expected learning rate. When the expected learning rate is relatively small, the game always stays in the first case. When the expected learning rate is intermediate, the game is initially in the third case, and switches to the first case if no player has received the Poisson signal for a sufficiently long time. When the expected learning rate is very high, the game starts out in the second case and eventually switches to the first case. Compared to the uninformed benchmark, learning causes longer (resp. shorter) delays when the learning rate is sufficiently low (resp. high).

In the good-news model, we conduct comparative statics analysis with respect to the learning rate. When the learning rate is very low, the average expected concession rate is determined by the indifference condition of the uninformed players, who become less optimistic about the concession payoffs as the learning rate goes up. Although an increase in the learning rate results in more informed players, who will concede immediately, the expected concession rate has to be lower in equilibrium in order to make the less optimistic uninformed players indifferent between conceding and staying. As a result, the expected delay is increasing in the learning rate. Due to this negative effect, an increase in the learning rate has no impact on welfare (in terms of a player's expected equilibrium payoff at the start of the game) when the learning rate is sufficiently low.

In contrast, when the learning rate is very high, a higher learning rate decreases the expected delay, and thus increases the welfare. Here, an increase in the learning rate has two opposite effects on the average expected concession rate. First, it leads to a higher average expected concession rate by increasing the chance of getting informed, since the informed players have the highest expected concession rate. Second, it leads to a lower average expected concession rate by making the uninformed players more reluctant to concede. When the learning rate is sufficiently high, a higher learning rate leads to a higher average expected concession rate by making the distribution of posterior beliefs more dispersed.

In the bad-news model, there is always one unique equilibrium pattern because the uninformed are more willing to concede than the informed players who have received bad-news signals. Initially, an uninformed player randomizes between conceding and staying, while an informed player strictly prefers staying. If both players receive the Poisson signal before conceding, the game eventually switches to a war of attrition between the informed players.

Different from the good-news model, the uninformed players obtain an additional value from receiving the Poisson signal in the bad-news model. In the good-news model, the continuation value after receiving the Poisson signal is always the high concession payoff, since the informed players always concede first. However, in the bad-news model, this continuation value is strictly larger than zero (the low concession payoff), since the informed benefit from the concession of the uninformed players. Because of this positive learning value, the uninformed players are more reluctant to concede. We show by example that in the bad-news model, learning leads to a longer expected delay compared to the uninformed benchmark.[2] Moreover, an increase in the learning rate has no impact on welfare in the

[2] In an earlier version of the paper, we show the opposite result when the positive learning value is absent.

bad-news model, since the uninformed players are always indifferent at the beginning of the game.

In the literature, continuous-time wars of attrition have been studied under both complete information ([8]) and incomplete information ([1], [6, 7], and [12]).[3] This paper extends the literature by considering a situation in which incomplete information is endogenously caused by learning. The game starts as a complete information game, and incomplete information about the payoffs develops over time.

The remainder of this paper is organized as follows. Section 2 presents the concession game. Section 3.1 and 3.2 analyze the uninformed benchmark and the full-information benchmark, respectively. Section 4 characterizes the symmetric equilibria of this war of attrition under the good-news model, and Section 5 discusses how the learning rate affects expected delay and welfare. Section 6 contains the equilibrium characterization and comparative statics results under the bad-news model. Section 7 concludes the paper.

2. The Game

2.1. Model Setting

Consider a continuous-time war of attrition with two risk-neutral players ($i = 1, 2$) without discounting. Both players decide when to concede. As long as neither player concedes, the game continues with each player incurring a flow cost c, which reflects the cost of delay. The game ends when one of the players concedes. The players' lump-sum payoffs in this event are specified in the following matrix:

		2: *Stay*	2: *Concede*
1	*Stay*	$-, -$	$v_H, \quad v_2$
	Concede	v_1, v_H	$M, \quad M$

If player i stays while her opponent $-i$ concedes, then player i is the winner of the game and gets a winning payoff of v_H. If player i concedes first, then she is the loser and gets a concession payoff v_i. The payoff when both players concede simultaneously is M. It is common knowledge that $M < v_H$, but there is incomplete information about concession payoffs v_1 and v_2. In particular, we assume that v_1 and v_2 are independently and identically drawn from a binary distribution: v_i can be either a positive number $v_L < v_H$ or zero. Each player i initially does not know the exact value of v_i. It is common knowledge that $v_i = v_L$ with prior probability p_0.

[3] The game is a special case of the more general quitting games in [14] and timing games in [10].

Following [9] and [15], we introduce learning by assuming that as long as no player concedes, each player receives an exogenous private signal which is distributed as the first arrival time of a Poisson process. The Poisson processes are independent across players, and for simplicity, we assume that the arrival of the Poisson signal completely resolves uncertainty about v_i. In the good-news model, the arrival rate is λ if $v_i = v_L$ and zero otherwise. After receiving this good-news signal, player i assigns probability one to the event that $v_i = v_L$. Absence of the signal will make the player increasingly pessimistic about the probability that $v_i = v_L$. In the opposite bad-news model, the arrival rate is λ if $v_i = 0$ and zero otherwise. After receiving this bad-news signal, player i assigns probability one to the event that $v_i = 0$. Absence of the signal will make the player increasingly optimistic about the probability that $v_i = v_L$.

2.2. Strategies and Equilibrium

At any point in time, a player is either informed or uninformed about her concession payoff. We refer to the former as an informed player, and the latter as an uninformed player.

A strategy for the uninformed player i is a cumulative distribution function $X^i : \mathbb{R}_+ \to [0, 1]$, where $X^i(t)$ denotes the probability that player i concedes to her opponent $-i$ by time t (inclusive).[4] Given $X^i(t)$, let γ_t^i be the joint probability of player i being uninformed at time t and not having conceded by time t. Obviously, $\gamma_0^i = 1$, and take the good-news model for example,

$$\gamma_t^i = \left(p_0 e^{-\lambda t} + 1 - p_0\right)\left(1 - X^i(t)\right), \tag{1}$$

where $p_0 e^{-\lambda t} + 1 - p_0$ is the probability that player i has not received the good-news signal. Hence, the evolution of γ_t^i is fully determined given $X^i(t)$. In particular, if $X^i(t)$ has a jump at time t, then γ_t^i drops at t; if $X^i(t)$ is continuously differentiable at time t, then uninformed player i's *concession rate* at t is computed as $x_t^i = \frac{dX^i(t)/dt}{1-X^i(t)}$. The probability of uninformed player i conceding to her opponent in the interval $[t, t+dt)$ is then $x_t^i dt$. Equation (1) implies that γ_t^i follows the law of motion

$$\dot{\gamma}_t^i = -x_t^i \gamma_t^i - \lambda q_t^i \gamma_t^i, \tag{2}$$

where q_t^i is the posterior probability that uninformed player i has the type to receive the Poisson signal. In the above good-news example, $q_t^i = \frac{p_0 e^{-\lambda t}}{p_0 e^{-\lambda t} + 1 - p_0}$ is the posterior probability that $v_i = v_L$ conditional on no signal being received; in the bad-news model, $q_t^i = \frac{(1-p_0)e^{-\lambda t}}{p_0 + (1-p_0)e^{-\lambda t}}$ is the posterior probability that $v_i = 0$ conditional on no signal being received. γ_t^i declines

[4] This definition of strategy avoids the well-known problem in continuous-time games in which well-defined strategies may not lead to well-defined outcomes, as shown by [2] and [13].

over time through two channels: concession at rate x^i_t, and arrival of the private signal at rate λq^i_t.

A strategy for the informed player i is a collection $(Y^i(\cdot;\tau))_{\tau\geq 0}$ of distribution functions $Y^i(\cdot;\tau):[\tau,\infty)\to[0,1]$, where τ is the time when the first Poisson signal is received.[5] Both $X^i(0)$ and $Y^i(\tau;\tau)$ are allowed to be strictly positive so that player i concedes to player $-i$ immediately. We use Z^i to denote the overall strategy of player i: $Z^i=(X^i,(Y^i(\cdot;\tau))_\tau)$. Z^i determines $F^i(t)$, which is the probability player $-i$ assigns to player i conceding by time t. The cumulative probability, $F^i(t)$, comes from concession by both informed and uninformed player. Take the good-news model for example, and we have:

$$1-F^i(t)=\gamma^i_t+p_0\int_0^t \lambda e^{-\lambda\tau}(1-X^i(\tau))(1-Y^i(t;\tau))d\tau. \tag{3}$$

The interpretation of equation (3) is as follows. If player i has not conceded by time t, she can be either uninformed or informed. The probability of the former event is γ^i_t, while the probability of the latter event is $p_0\int_0^t \lambda e^{-\lambda\tau}(1-X^i(\tau))(1-Y^i(t;\tau))d\tau$. Notice that the term $(1-X^i(\tau))$ is needed in the expression because an uninformed player has to stay in the game in order to receive the good-news signal.

From equation (1), equation (3) can be rewritten as:

$$F^i(t)=1-\gamma^i_t-\int_0^t \lambda q^i_\tau\gamma^i_\tau\left(1-Y^i(t;\tau)\right)d\tau.$$

If $X^i(t)$ is continuously differentiable, then equation (2) implies

$$F^i(t)=\int_0^t\left[x^i_\tau\gamma^i_\tau+\lambda q^i_\tau\gamma^i_\tau Y^i(t;\tau)\right]d\tau, \tag{4}$$

where $\int_0^t x^i_\tau\gamma^i_\tau d\tau$ is the probability player $-i$ assigns to uninformed player i conceding by time t and $\int_0^t \lambda q^i_\tau\gamma^i_\tau Y^i(t;\tau)d\tau$ is the probability player $-i$ assigns to informed player i conceding by time t. If $F^i(t)$ is continuously differentiable at time t, then $f^i_t=\frac{dF^i(t)/dt}{1-F^i(t)}$ is player i's *expected concession rate* at time t conditional on no concession before time t. Upon reaching time t, player $-i$ then expects player i to concede in the interval $[t,t+dt)$ with probability $f^i_t dt$.

[5] There is a continuum of informed players that is indexed by the arrival time of the first Poisson signal. The informed players are not required to use the same strategy at any time t. There might be a continuum of equilibria by assigning different informed players different concession rates. However, all of the equilibria are outcome equivalent in terms of the expected concession rate.

Given Z^{-i}, an informed player i's expected payoff by conceding at time t is

$$\begin{aligned} W^i(t;\tau,Z^{-i}) &= \int_{\tau\le s<t} (v_H - c(s-\tau))\,dF^{-i}(s|\tau) \\ &+ (M - c(t-\tau))\left(F^{-i}(t|\tau) - F^{-i}(t-|\tau)\right) \\ &+ (v_i - c(t-\tau))\left(1 - F^{-i}(t|\tau)\right), \end{aligned} \tag{5}$$

where τ is the time when the first Poisson signal is received, $F^{-i}(t|\tau)$ is the probability player i assigns to player $-i$ conceding by time t conditional on time τ having been reached, $F^{-i}(t-|\tau) = \lim_{s\nearrow t} F^{-i}(s|\tau)$, and v_i is informed player i's concession payoff.[6]

Given Z^{-i}, an uninformed player i's expected payoff from conceding at time t is

$$V^i(t;Z^{-i}) = \mathbb{E}_\tau\Big[\Pr(\tau\le t)V_I^i(\tau;Z^{-i}) + \Pr(\tau>t)V_U^i(t;Z^{-i})\Big], \tag{6}$$

where V_I^i is player i's expected payoff if she gets informed at $\tau \le t$, and V_U^i is player i's expected payoff if she has not become informed by t. V_I^i depends on τ but not on t, because it is not necessarily the best response for an informed player to concede at t. Let $\hat{W}^i(\tau;Z^{-i}) = \sup_{t\ge\tau} W^i(t;\tau,Z^{-i})$, and p_t be uninformed player i's posterior belief that $v_i = v_L$. Then,

$$V_I^i(\tau;Z^{-i}) = \int_{s<\tau} (v_H - cs)\,dF^{-i}(s) + (v_H - c\tau)\left(F^{-i}(\tau) - F^{-i}(\tau-)\right) + \hat{W}^i(\tau;Z^{-i})\left(1 - F^{-i}(\tau)\right),$$

and

$$V_U^i(t;Z^{-i}) = \int_{s<t} (v_H - cs)\,dF^{-i}(s) + (M - ct)\left(F^{-i}(t) - F^{-i}(t-)\right) + (p_t v_L - ct)\left(1 - F^{-i}(t)\right).$$

A strategy profile (Z^1, Z^2) is an *equilibrium* if the expected payoffs of both the uninformed and the informed players are maximized at any time t given the opponent's strategy. Throughout this paper, we will focus on symmetric equilibrium in which $X^1(t) = X^2(t)$ and $Y^1(t;\tau) = Y^2(t;\tau)$ for all $t \ge 0$ and $\tau \le t$. The superscript i will be omitted in the future if no confusion arises.

3. War of Attrition without Learning

We will discuss two benchmarks in this section. In the first benchmark, both players are uninformed of their value of conceding until they actually concede. In the second benchmark, both players are privately informed of this value from the outset. These two benchmarks

[6] In the good-news model, $v_i = v_L$, and in the bad-news model, $v_i = 0$.

can be viewed as the limiting cases where $\lambda = 0$ and ∞, respectively.

3.1. Benchmark I: Both Players Uninformed

In the uninformed benchmark, each player i assigns probability p_0 to $v_i = v_L$ throughout the game. As a result, no matter when concession occurs, the winner gets v_H while the loser of the game gets the expected payoff $p_0 v_L$. The unique symmetric equilibrium is characterized by the following proposition.[7] The proofs of the following and all subsequent results can be found in the appendix.

Proposition 1. *In the uninformed benchmark, the unique symmetric equilibrium is such that at any time $t \in [0, \infty)$, each player concedes at the constant rate $\frac{c}{v_H - p_0 v_L}$.*

In a symmetric equilibrium, no player concedes with a positive probability at time 0, and each player is always indifferent between conceding and staying at any point in time. By conceding, each player receives $p_0 v_L$ for certain, whereas, by staying, each player pays flow cost c and expects her opponent to concede at rate f_t. Therefore, the best responses of the players are determined by the sign of $f_t(v_H - p_0 v_L) - c$. Each player i strictly prefers to stay if the expected concession rate f_t is strictly larger than $\frac{c}{v_H - p_0 v_L}$; is indifferent between conceding and staying if f_t is exactly $\frac{c}{v_H - p_0 v_L}$; and strictly prefers to concede if f_t is strictly smaller than $\frac{c}{v_H - p_0 v_L}$. In the unique symmetric equilibrium, the equilibrium concession rate is $\frac{c}{v_H - p_0 v_L}$ to make each player always indifferent between conceding and staying. Although the game may continue for an arbitrarily long period of time, the expected delay is finite in equilibrium, and equals $\frac{v_H - p_0 v_L}{2c}$.

3.2. Benchmark II: Both Players Informed

In the second benchmark, there is immediate revelation of payoffs. Each player i knows exactly what v_i is, but her opponent does not. The initial beliefs are such that each player has high concession payoff v_L with probability p_0, and low concession payoff 0 with probability $1 - p_0$. This game is similar to the war of attrition with incomplete information analyzed by [1], [7], and [12]. The next result shows that, in this benchmark, the unique symmetric equilibrium starts with the high-value player randomizing and the low-value player strictly preferring to stay; eventually, as the posterior probability of v_L reaches zero, the low-value player randomizes until the end of the game. Let $t' = \left(\frac{v_H - v_L}{c}\right) \log\left(\frac{1}{1-p_0}\right)$.

[7] In a complete-information war of attrition, there always exists a degenerate equilibrium in which player i concedes immediately, while player $-i$ never concedes, as shown by [11]. However, as shown by [1], the equilibrium is unique if there is uncertainty about the players' types. The symmetric equilibrium in a complete-information war of attrition corresponds to the limit of the unique equilibrium in an incomplete-information war of attrition as the degree of type uncertainty converges to 0.

Proposition 2. *In the full-information benchmark, the unique symmetric equilibrium has the following properties:*

(1) For $t \in [0, t')$*, the high-value player concedes at a rate* $\frac{c/(v_H - v_L)}{1-(1-p_0)^{\frac{ct}{v_H - v_L}}}$*, while the low-value player never concedes;*

(2) for $t \in [t', \infty)$*, only the low-value players stay, and they concede at a constant rate* $\frac{c}{v_H}$*;*

(3) for any $p_0 \in (0, 1)$*, the expected delay is smaller than in the uninformed benchmark.*

The equilibrium analysis follows that in [1], [7], and [12]. In a symmetric equilibrium, no player will concede with a positive probability at any time t, and $F^i(t)$ is continuous and strictly increasing. The best responses of the high-value (resp. low-value) players are determined by the sign of $f_t(v_H - v_L) - c$ (resp. $f_t v_H - c$). Hence, high-value (resp. low-value) player i strictly prefers to stay if the expected concession rate f_t is strictly larger than $\frac{c}{v_H - v_L}$ (resp. $\frac{c}{v_H}$); is indifferent between conceding and staying if f_t is exactly $\frac{c}{v_H - v_L}$ (resp. $\frac{c}{v_H}$); and strictly prefers to concede if f_t is strictly smaller than $\frac{c}{v_H - v_L}$ (resp. $\frac{c}{v_H}$). Obviously, the high-value player is more willing to concede; hence, in the unique symmetric equilibrium, the game starts as a war of attrition between the high-value players, while the low-value player strictly prefers to stay. The prior of being a high-value player is p_0, and the posterior, $1 - (1 - p_0)e^{\frac{ct}{v_H - v_L}}$, is decreasing over time. Eventually, as the high-value player has conceded with probability one ($t \geq t'$), the expected concession rate drops to $\frac{c}{v_H}$ and the game switches to a war of attrition between the low-value players.

Proposition 2 also shows that the expected delay is smaller than that in the uninformed benchmark. This result has a very intuitive explanation. In the full-information benchmark, with probability p_0, $v_i = v_L$ and the expected concession rate is $\frac{c}{v_H - v_L}$; with probability $1 - p_0$, $v_i = 0$ and the expected concession rate is $\frac{c}{v_H}$. Since $\frac{c}{v_H - p v_L}$ is convex in p, we have $p_0 \frac{c}{v_H - v_L} + (1 - p_0)\frac{c}{v_H} > \frac{c}{v_H - p_0 v_L}$, so the average expected concession rate is higher in the full-information benchmark. This effect is strong enough to outweigh the fact that in the full-information benchmark the uninformed type starts conceding only after some fixed amount of time.

4. War of Attrition with Good-News Learning

In this section, we will first characterize equilibrium in the good-news model. In this model, at any time $t > 0$, the informed player believes $v_i = v_L$ for certain, while the uninformed player is still unsure about her concession payoff. Recall that p_t denotes the posterior probability which an uninformed player assigns to the concession payoff v_L at time

t. From Bayes rule, $p_t = \frac{p_0 e^{-\lambda t}}{p_0 e^{-\lambda t}+1-p_0}$, since the high-value player receives no signal before time t with probability $e^{-\lambda t}$ while the low-value player receives no signal for sure.

The game starts as a war of attrition with complete information. However, incomplete information develops over time due to learning. Any candidate equilibrium shares the main features of a war of attrition with incomplete information studied in [1], [7], and [12]. Therefore, as in the full-information benchmark, in a symmetric equilibrium, no player will concede with positive probability at any time t, and $F^i(t)$ is continuous and strictly increasing. This enables us to focus on analyzing the expected concession rate f_t^i.

For both the informed and uninformed players, there are three possibilities: strictly prefer conceding, strictly prefer staying, or indifference. Thus, there are nine different combinations in total. The next lemma shows that only three of them can occur in equilibrium.

Lemma 1. *In any symmetric equilibrium, at any time t, only one of the following three cases is possible:*

(1) the uninformed player is indifferent between conceding and staying and the informed player strictly prefers conceding;

(2) the informed player is indifferent between conceding and staying and the uninformed player strictly prefers staying;

(3) the informed player strictly prefers conceding but the uninformed player strictly prefers staying.

We shall refer to case (1) as 'uninformed indifference', to case (2) as 'informed indifference', and to case (3) as 'strict preferences'. The above lemma has very intuitive interpretations. Since the informed player is more optimistic about the private payoff state than the uninformed player, the informed player has a higher incentive to concede. As a result, if the uninformed player is indifferent between conceding and staying, the informed player must strictly prefer conceding; if the informed player is indifferent between conceding and staying, the uninformed player must strictly prefer staying. In the full-information benchmark, at each point in time, either the high-value or the low-value player is indifferent between conceding and staying. Here, however, it is possible that neither the uninformed nor the informed player is indifferent.

Based on Lemma 1, Sections 4.1-4.3 will fully characterize symmetric equilibrium behavior in the good-news model. Throughout these sections, we assume that $v_H \geq 2p_0 v_L$, which guarantees that in the absence of good news, $p_t(v_H - p_t v_L)$ decreases monotonically over time. Section 4.4 will briefly discuss the case where $v_H < 2p_0 v_L$.

4.1. Slow Learning

We will first characterizes the unique symmetric equilibrium when the expected learning rate is sufficiently low.

Proposition 3. *If* $\lambda p_0 \leq \frac{c}{v_H - p_0 v_L}$*, the unique symmetric equilibrium has the following properties: An uninformed player concedes with probability zero at time 0 and at a positive rate* $x_t = \frac{c}{v_H - p_t v_L} - \lambda p_t$ *for any* $t > 0$*. An informed player concedes with probability one upon receiving the first Poisson signal.*

For sufficiently small λ, the unique symmetric equilibrium thus always features uninformed indifference. In fact, the low expected arrival rate of good news makes it impossible for uninformed players to strictly prefer staying: even if informed players concede immediately, the expected rate λp_t at which a player becomes informed is too low for an uninformed opponent to strictly benefit from staying in the game.

For uninformed indifference, the expected concession rate must be $f_t = \frac{c}{v_H - p_t v_L}$. We provide a detailed proof of this result in the appendix. Intuitively, the uninformed player's value of staying, $V(p_t)$, must equal her value of conceding, $p_t v_L$, and it must satisfy the differential equation

$$-c + \lambda p_t \left(v_L - V(p_t)\right) + f_t \left(v_H - V(p_t)\right) - \lambda p_t (1 - p_t) V'(p_t) = 0. \quad (7)$$

Indeed, as long as an uninformed player pays cost c to stay, her opponent concedes at the rate f_t; in this event, her continuation value jumps to v_H. With arrival rate λp_t, the uninformed player receives good news and then strictly prefers conceding, so in this event her continuation value jumps to v_L. If the opponent does not concede and there is no news, finally, the uninformed player's value of staying decreases deterministically; this is captured by the term $-\lambda p_t(1 - p_t)V'(p_t)$.[8] Inserting $V(p_t) = p_t v_L$, we see that the last two terms cancel, leaving $f_t(v_H - p_t v_L) = c$.[9] Consequently, the concession rate of an uninformed player is the difference between f_t and λp_t, the concession rate generated by the arrival of good news.

Even if the condition $\lambda p_0 \leq \frac{c}{v_H - p_0 v_L}$ is not satisfied, as t tends to ∞, p_t decreases to zero. Therefore, for any λ, the inequality $\lambda p_t \leq \frac{c}{v_H - p_t v_L}$ will hold for t sufficiently large. Consequently, uninformed indifference and behavior as described in Proposition 3 will eventually arise in any symmetric equilibrium.

[8]As shown by [9], the law of motion for p_t satisfies: $\dot{p}_t = -\lambda p_t(1 - p_t)$.

[9]Intuitively, this is because the value $V(p_t)$ is linear in p_t, which is a martingale.

4.2. Intermediate Speed of Learning

For λp_0 somewhat higher than assumed in Proposition 3, uninformed indifference becomes impossible at the start of the game: the expected rate at which a player becomes informed (and then immediately concedes) makes it strictly beneficial for an uninformed player to stay. On the other hand, from the analysis of the full-information benchmark, we know that for informed indifference, the expected concession rate must be $\frac{c}{v_H - v_L}$. So, if $\lambda p_0 \in (\frac{c}{v_H - p_0 v_L}, \frac{c}{v_H - v_L}]$, there are strict preferences initially. Let T_1 satisfy:

$$T_1 = \frac{1}{\lambda} \log \left[\frac{p_0(1 - p_{T_1})}{(1 - p_0) p_{T_1}} \right],$$

where

$$p_{T_1} = \frac{\lambda v_H - \sqrt{\lambda^2 v_H^2 - 4\lambda c v_L}}{2\lambda v_L}.$$

Proposition 4. *If $\lambda p_0 \in (\frac{c}{v_H - p_0 v_L}, \frac{c}{v_H - v_L}]$, the unique symmetric equilibrium has the following properties: An informed player always concedes immediately, while an uninformed player never concedes prior to time T_1. For $t \geq T_1$, players behave as described in Proposition 3.*

For expected learning rates in an intermediate range, the expected concession rate equals the expected arrival rate of the first Poisson signal, λp_t. Strict preferences further imply that if the game has not ended yet, it is common knowledge among the players that neither of them is informed. When T_1 is reached, therefore, continuation play can indeed be the one described in Proposition 3, started at the belief p_{T_1}.

4.3. Fast Learning

For λp_0 even higher than assumed in Proposition 4, informed players have to be indifferent at the start of the game. The high expected arrival rate of good news makes it impossible for informed players to strictly prefer conceding, because otherwise the expected concession rate is so high that the opponent strictly benefits from staying in the game. Let T be implicitly defined by

$$p_0(1 - e^{-\lambda T}) = 1 - e^{-\frac{cT}{v_H - v_L}}. \tag{8}$$

Proposition 5. *If $\lambda p_0 > \frac{c}{v_H - v_L}$, all symmetric equilibria are outcome-equivalent and have the following properties:*[10]

[10] There is an indeterminacy in the informed player's strategy $X(t;\tau)$. Any strategy can be an equilibrium as long as equation (4) is satisfied.

(1) Up to time T, the informed player is indifferent between conceding and staying, while the uninformed player never concedes. The expected concession rate is $\frac{c}{v_H - v_L}$.

(2) If $v_H \geq 2v_L$, there exists a unique $\hat{\lambda} > \frac{c}{p_0(v_H - v_L)}$ such that the following holds: when $\lambda \leq \hat{\lambda}$, then $\lambda p_T \geq \frac{c}{v_H - p_T v_L}$ and players behave as described in Proposition 4 for $t \geq T$; when $\lambda < \hat{\lambda}$, then $\lambda p_T < \frac{c}{v_H - p_T v_L}$ and players behave as described in Proposition 3 for $t \geq T$.[11]

(3) $\hat{\lambda}$ is decreasing in p_0.

In equilibrium, an informed player randomizes until the posterior belief of being an informed player reaches zero. An informed player concedes with probability one by time T, which is determined by equation (8). The left-hand side of equation (8), $p_0(1 - e^{-\lambda T})$, is the probability that a player has received good news by time T; the right-hand side, $1 - e^{-\frac{cT}{v_H - v_L}}$, is the probability that a player has conceded until time T, given that the expected concession rate is $\frac{c}{v_H - v_L}$.

In summary, a symmetric equilibrium can be completely characterized in terms of expected rates of learning λp_0. Players behave as described in Proposition 3 (resp. 4) for small (resp. intermediate) expected rates of learning. For high expected rates of learning, any symmetric equilibrium features informed indifference until T defined by equation (8). If the game has not ended yet at T, play continues under common knowledge that no player is informed. Depending on the value of λ, the continuation play switches to either strict preferences or uninformed indifference. In the former case (when λ is smaller than $\hat{\lambda}$), there exists T_1 such that the symmetric equilibrium features strict preferences for $t \in (T, T + T_1)$; and uninformed indifference for $t \in (T + T_1, \infty)$. In the latter case (when λ larger than $\hat{\lambda}$), the symmetric equilibrium always features uninformed indifference after T. The expected concession rate may not be continuous in time (as shown by the full-information benchmark), but it is non-increasing over time for any value of λ: while it is relatively easy to reach an agreement initially, it becomes increasingly difficult over time.[12]

The equilibrium characterization is more complicated if $v_H < 2p_0 v_L$, because there may exist two different roots $0 < p_2 < p_1 < 1$ solving $p(v_H - pv_L) = \frac{c}{\lambda}$. Therefore, in a symmetric equilibrium, it is possible that a symmetric equilibrium features uninformed indifference when $p < p_2$ or $p > p_1$, and strict preferences when $p \in (p_2, p_1)$. Although the equilibrium can be characterized similarly, we skip the analysis since it provides no new insights into the

[11]From the proof in the appendix, $v_H \geq 2v_L$ is actually stronger than what is needed. It is enough to assume $v_H \geq (1 + p_0)v_L$.

[12]From the equilibrium characterizations (Propositions 3-5), the expected concession rate is a constant $\frac{c}{v_H - v_L}$ for $t \leq T$, and there must be a drop in the expected concession rate if the game has not ended at T. The expected concession rate decreases continuously for $t > T$. Therefore, the expected concession rate is non-increasing over time.

comparative statics.

5. Comparative Statics with Respect to the Learning Rate

Based on the equilibrium characterization in the previous section, this section will discuss how the learning rate λ affects expected delay and welfare in equilibrium.

5.1. Expected Delay

In a symmetric equilibrium, the expected concession rate changes over time. As p_t decreases, it becomes increasingly difficult to reach an agreement. The non-stationary path of beliefs and concession rates makes it difficult to derive explicit expressions for the expected delay. However, we show that an increase in the learning rate increases the expected delay when λ is relatively low, but decreases the expected delay when λ is relatively high.

Proposition 6. *The expected delay in symmetric equilibrium is non-monotonic in λ. If $\lambda p_0 \leq \frac{c}{v_H - p_0 v_L}$, the expected delay is increasing in λ. Moreover, the expected delay is decreasing in λ when λ is sufficiently large.*

The non-monotonicity result is driven by the fact that learning rate λ affects the expected concession rate in two opposite ways. On the one hand, faster learning makes it more likely for an uninformed player to receive good news, and this increases the expected concession rate, since an informed player is more willing to concede. On the other hand, faster learning makes an uninformed player more willing to wait for both good news and her opponent's concession, and this strategic response leads to a lower expected concession rate. For small expected rates of learning, this strategic effect dominates, because in equilibrium, the expected concession rate is determined to make an uninformed player indifferent between conceding and staying, and an increase in λ induces the uninformed player to wait longer. For intermediate expected rates of learning, an uninformed player will always stay while an informed player will always concede. Hence, faster learning leads to a quicker concession. For high expected rates of learning, an increase in λ does not affect the expected concession rate in equilibrium, which is always $\frac{c}{v_H - v_L}$ to make an informed player indifferent.

Naturally, when $\lambda p_0 \leq \frac{c}{v_H - p_0 v_L}$, the equilibrium will always feature uninformed indifference. Although an increase in λ leads to a larger measure of informed players who will concede immediately, this increase in the concession rate is completely crowded out by the decrease in the concession rate of uninformed players, so as to make the less optimistic uninformed players indifferent between conceding and staying.

It is difficult to derive an explicit comparative static result for $\lambda p_0 > \frac{c}{v_H - p_0 v_L}$, because of the coexistence of the opposite effects. However, Proposition 6 shows that the expected

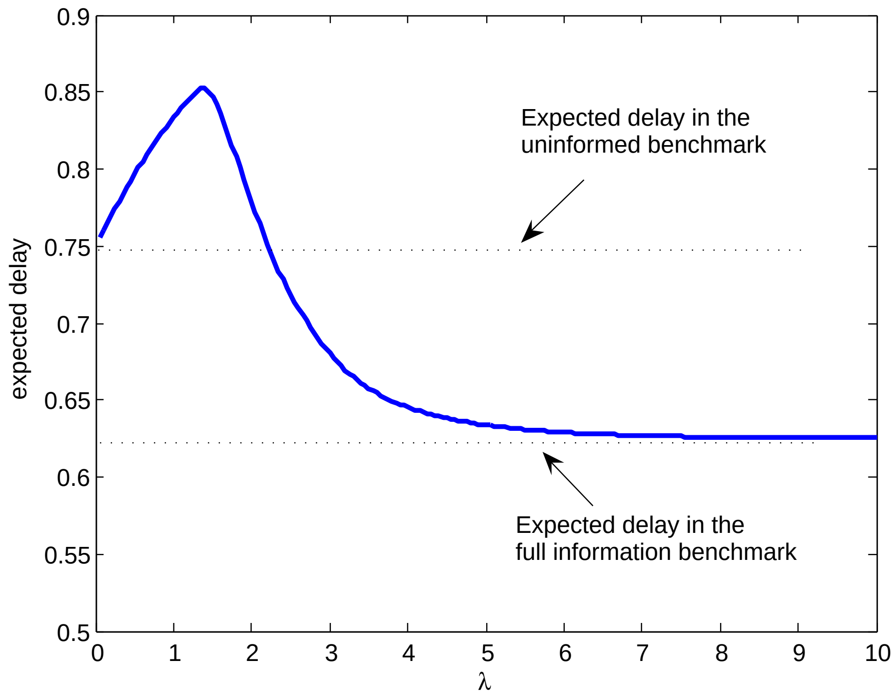

Figure 1: Expected delay as a function of λ ($v_H = 2, v_L = c = 1, p_0 = 0.5$)

delay is decreasing in λ when λ is large. Then, the equilibrium consists of two phases: one features informed indifference and the other features uninformed indifference. An increase in λ has three effects on the average expected concession rate in this situation. First, when λ becomes higher, the game has a higher probability to stop before T defined by equation (8). Second, a higher λ leads to a lower posterior belief p_T: the uninformed players become less optimistic. Finally, a higher λ leads to a longer delay for $t \geq T$. The comparison of the two benchmarks suggests that the expected delay is decreasing in λ if the third effect does not exist. As $\frac{c}{v_H - pv_L}$ is convex in p, a higher λ may lead to a higher expected concession rate by making the distribution of p more dispersed.[13] When λ approaches ∞, the third effect becomes a second order effect, because as p_T goes to zero, the possible reduction in the expected concession rate for $t \geq T$ is negligible. As a result, the expected delay will decrease in λ when λ is sufficiently large.

[13]To be more specific, assume that there is no learning once the game reaches T, and the players concede with probability x before T. Then the expected concession rate can be written as:

$$x\frac{c}{v_H - v_L} + (1-x)\frac{c}{v_H - \frac{p_0 - x}{1-x}v_L},$$

where $\frac{p_0 - x}{1-x}$ is the posterior belief p_T. The above term is decreasing in x, which in turn is increasing in λ.

Based on Proposition 6, it is natural to conjecture that the expected delay is first increasing and then decreasing in λ. Figure 1 confirms this for a numerical example. Compared to the uninformed benchmark, learning does not necessarily lead to shorter delays, and this occurs only when the learning rate λ is sufficiently high.

5.2. Welfare

Based on the above equilibrium analysis, we compute the expected equilibrium payoff of each player.

First, for small expected rates of learning, if the common belief is that $v = v_L$ occurs with probability p_t, then the uninformed player i's value is $V(p_t) = p_t v_L$ since she is indifferent between conceding and staying. Second, for intermediate expected rates of learning, the uninformed player's value $V(p_t)$ satisfies the differential equation

$$2\lambda p_t V(p_t) = -c + \lambda p_t(v_H + v_L) - \lambda p_t(1 - p_t)V'(p_t).$$

By paying flow cost c to stay, an uninformed player receives good news with arrival rate λp_t, and her continuation value jumps to v_L in this event. With arrival rate λp_t. her opponent receives good news and concedes; in this event, her continuation value jumps to v_H. The differential equation has the general solution

$$V(p) = \frac{1}{2}(v_L + v_H - \frac{c}{\lambda}) - \frac{c}{\lambda}(1-p)^2 \log(\frac{p}{1-p}) - \frac{c}{\lambda}(1-p) + D_1(1-p)^2,$$

where the coefficient D_1 is determined by the boundary condition that $V(p) = pv_L$ when $\lambda p(v_H - pv_L) = c$.

Finally, for intermediate expected rates of learning, the uninformed player's value $V(p_t)$ satisfies the differential equation,

$$(f_t + \lambda p_t)V(p_t) = -c + \lambda p_t v_L + f_t v_H - \lambda p_t(1 - p_t)V'(p_t).$$

This equation differs from the previous one in that the opponent's concession rate is f_t rather than λp_t. From the indifference condition of the informed player, $f_t v_H - c = f_t v_L$, the differential equation is rewritten as

$$(\frac{c}{v_H - v_L} + \lambda p_t)V(p_t) = (\frac{c}{v_H - v_L} + \lambda p_t)v_L - \lambda p_t(1 - p_t)V'(p_t)$$

with the general solution

$$V(p) = v_L + D_2(\frac{1-p}{p})^{\frac{c}{\lambda(v_H - v_L)}}(1 - p).$$

The boundary condition is given by value matching at p_T, where T is defined by equation

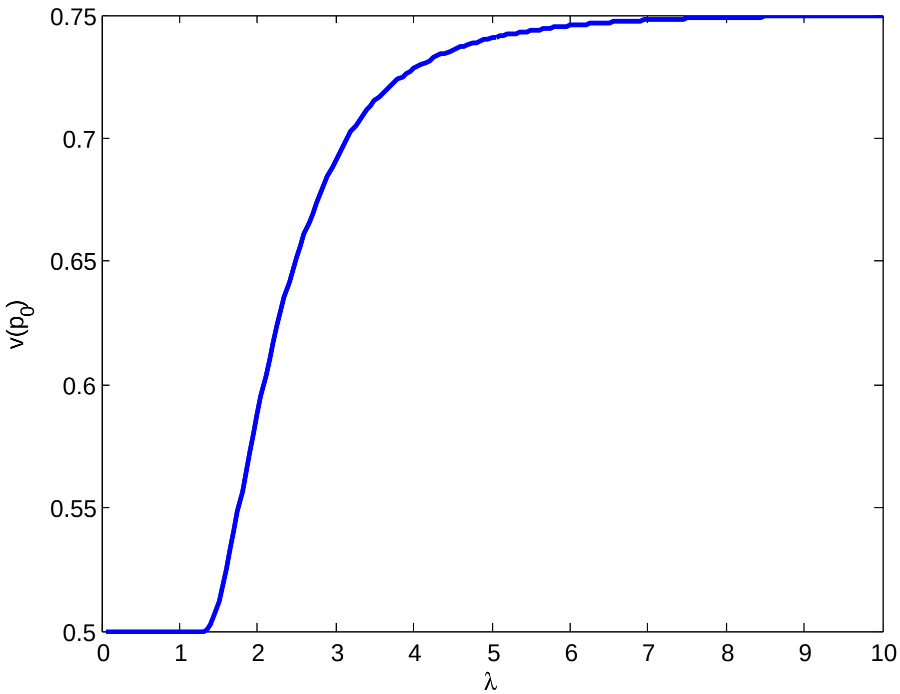

Figure 2: Welfare as a function of λ ($v_H = 2, v_L = c = 1, p_0 = 0.5$)

(8). Based on the above expressions for the value functions, it can be shown that welfare is increasing in λ when λ is sufficiently high.

Corollary 1. *Suppose λ is sufficiently high and, therefore, the equilibrium is as described in Proposition 5. Then,* $\frac{\partial V(p_0;\lambda)}{\partial \lambda} > 0$.

Figure 2 provides an illustration of the equilibrium value $V(p_0)$ as a function of λ. As expected, increasing λ has a positive effect on $V(p_0)$ only for intermediate and high expected rates of learning; for small expected rates of learning, increasing λ has no impact on the equilibrium value $V(p_0) = p_0 v_L$, because faster learning has both a positive effect by shortening the expected time of receiving good news and a negative effect of increasing delay. It turns out that these two opposite effects cancel each other out at time 0; hence, increasing λ has no impact on the expected equilibrium payoff. In contrast, when λ is so high that the expected delay is decreasing in λ, the negative effect is overturned and naturally, welfare increases in λ.

6. War of Attrition with Bad-News Learning

In this section, we will consider the bad-news model in which receiving a Poisson signal immediately reveals $v = 0$. In this model, the posterior probability which an uninformed player assigns to the concession payoff v_L at time t satisfies $p_t = \frac{p_0}{p_0+(1-p_0)e^{-\lambda t}}$: an uninformed player is more optimistic about the concession payoff than an informed player, and, hence, is more willing to concede. As a result, the unique symmetric equilibrium always features two phases for any $\lambda > 0$: in the first phase, the uninformed players are randomizing between conceding and staying, while the informed players strictly prefer staying; in the second phase, only the informed players are left in the game and they are randomizing between conceding and staying. Let T be implicitly defined by

$$F(T)p_T - \int_0^T \lambda p_t(1-p_t)\Sigma(t)dt = p_0, \tag{9}$$

where

$$F(T) = \frac{\int_0^T c\Gamma(t)dt}{\int_0^T \lambda v_L p_t(1-p_t)\Gamma(t)dt + (v_H - p_T v_L)\Gamma(T)}, \tag{10}$$

$$\Sigma(t) = \frac{\int_0^t c\Gamma(s)ds - F(T)\int_0^t \lambda v_L p_s(1-p_s)\Gamma(s)ds}{(v_H - p_t v_L)\Gamma(t)}, \tag{11}$$

and

$$\Gamma(t) = e^{\frac{ct}{v_H}} \left[\frac{(v_H - v_L)p_0 e^{\lambda t} + (1-p_0)v_H}{v_H - p_0 v_L}\right]^{\frac{cv_L}{\lambda v_H(v_H - v_L)}}. \tag{12}$$

Proposition 7. *The unique candidate symmetric equilibrium in the bad-news model has the following properties:*

(1) Each uninformed player concedes with probability zero at time 0 and at a positive rate between time 0 and time T. An informed player never concedes before T.

(2) After time T, only informed players stay and they concede at rate $\frac{c}{v_H}$.

In contrast to the good-news model, there is one unique equilibrium pattern in the bad-news model. Uninformed players always concede before informed players. For any $t < T$, an uninformed player's value of staying, $V(p_t)$, must equal her value of conceding, $p_t v_L$, and it must satisfy the differential equation

$$-c + \lambda(1-p_t)(W(p_t) - V(p_t)) + f_t(v_H - V(p_t)) + \lambda p_t(1-p_t)V'(p_t) = 0. \tag{13}$$

Recall that in the good-news model, an informed player's value, $W(p_t)$, is v_L since the informed player concedes immediately. However, in the bad-news model, $W(p_t)$ is not zero since the informed player never concedes before T. In particular, $W(p_t)$ must satisfy the differential equation

$$-c + f_t(v_H - W(p_t)) + \lambda p_t(1-p_t)W'(p_t) = 0. \tag{14}$$

Using $V(p_t) = p_t v_L$, we solve $W(p_t) = \frac{F(T)-F(t)}{1-F(t)} p_t v_L > 0$. This implies that the equilibrium concession rate is

$$f_t = \frac{c - \lambda p_t(1-p_t)v_L(F(T)-F(t))/(1-F(t))}{v_H - p_t v_L}. \tag{15}$$

The equilibrium concession rate consists of two parts. The first part, $\frac{c}{v_H - p_t v_L}$ is increasing over time: as an uninformed player becomes increasingly optimistic about the concession payoff, she is more willing to concede. However, receiving bad news also brings positive learning value $W(p_t)$, which makes an uninformed player reluctant to concede. This is captured by the negative part of equation (15). The negative part gradually converges to zero as t approaches T.

We derive $F(t)$ by solving the differential equation (15). The cutoff time T is pinned down by the fact that the uninformed players concede with probability one by time T. Equation (9), which determines T, is in turn derived from equation

$$p_0 = \int_0^T p_t F'(t)dt = p_T F(T) - \int_0^T \lambda p_t(1-p_t)F(t)dt. \tag{16}$$

p_0 is the measure of high-value players; while $\int_0^T p_t F'(t)dt$ is the measure of high-value players who have conceded until time T, which is the integral of the density of high-value players conceding at time $t \leq T$. These two terms have to be the same since only the low-value players stay after time T.

Compared to the uninformed benchmark, learning affects expected delay through three different channels. First, for $t < T$, learning reduces expected delay by making the uninformed players optimistic about the concession payoff. Second, for $t < T$, learning increases expected delay due to the positive learning value. Finally, for $t \geq T$, learning increases expected delay since the low value players have the lowest concession rate. Figure 3 illustrates a numerical example using the same parameter values as for Figure 1. For these parameters, the expected delay is actually monotonically increasing in λ.

As to welfare, since in equilibrium an uninformed player is always indifferent between conceding and staying, the equilibrium value $V(p_0)$ is always $p_0 v_L$. Therefore, in the bad-news model, although the learning rate would affect the expected delay, the equilibrium value $V(p_0)$ is not affected by the learning rate at all. On the one hand, faster learning makes it more likely to receive bad news, which reduces the expected payoff to 0. On the other hand, faster learning makes the uninformed opponent more willing to concede, and this leads to a higher expected payoff. In equilibrium, these two opposite effects cancel out at time 0; thus the equilibrium value does not change with the learning rate.

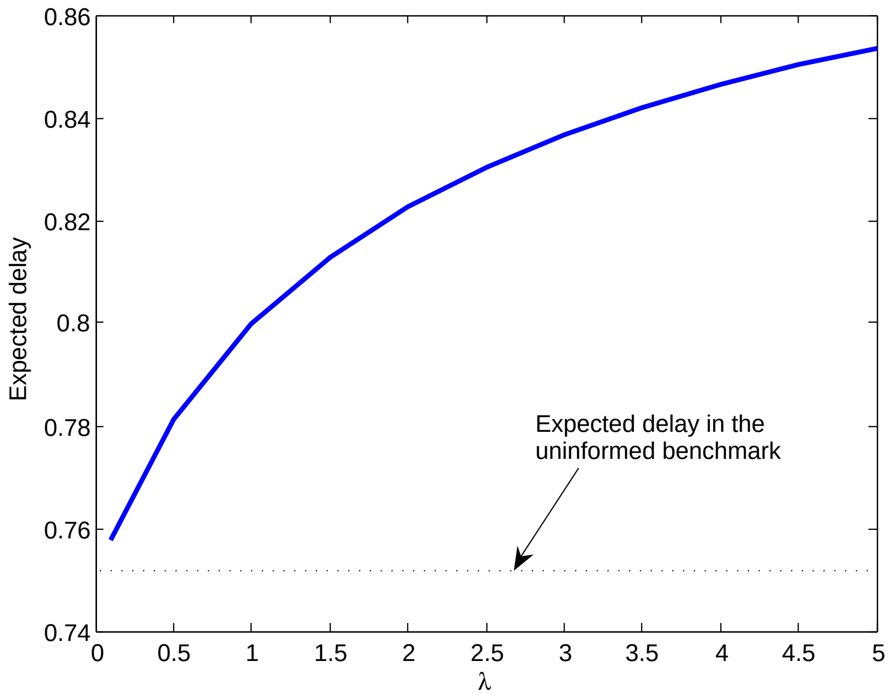

Figure 3: Expected delay as a function of λ ($v_H = 2, v_L = c = 1, p_0 = 0.5$)

7. Conclusion

Delay is a pervasive phenomenon in many realistic situations including bargaining or coordination. The paper considers a continuous-time war of attrition to investigate how exogenous learning about private payoffs affects delays in reaching agreements. In the model, the concession payoff can be either high or low, and at each point in time, players may receive a private Poisson signal that completely reveals the concession payoff to be high (good-news learning) or low (bad-news learning). In the good-news model, learning can cause shorter delays in reaching agreements, but not always. In particular, an increase in the learning rate causes longer delays when the learning rate is sufficiently low. In the bad-news model, we show by example that learning causes longer delays, but has no impact on welfare.

The model studied in this paper can be extended in several different ways. First of all, it would be interesting to explore a specification in which the signal follows a Brownian motion with drift v_i. Then the posterior belief follows a diffusion process, as shown in [4]. It is natural to conjecture an equilibrium in which there exists a cutoff $\bar{p}_t$ at time t such that players with $p_t \geq \bar{p}_t$ ($p_t < \bar{p}_t$) will concede (stay). However, analyzing the cutoff $\bar{p}_t$ is challenging. Second, in the model, it is assumed for simplicity that the two players are facing the same learning rate. This enables us to consider symmetric equilibria, in which no player concedes with a positive probability at time 0. It is more complicated to analyze equilibria for differential learning rates. In this case, as shown by [1], it is still possible to construct a non-degenerate equilibrium in which concession may continue for an arbitrarily long period of time. However, one of the two players will concede with a positive probability at time 0. Finally, endogenous information acquisition can be added into the model, by assuming that the players can increase the learning rate through costly effort as modeled by [5]. At every instant of time, both players have to decide whether or not to concede, and the learning rate if they choose to stay. In the working paper version of this article, it is shown that in the good-news model, no player will acquire information if the maximum achievable learning rate is lower than $\frac{c}{p_0(v_H - p_0 v_L)}$. More generally, one would expect that there is insufficient incentive to acquire information in this setting.

Appendix

A.1 Proof of Proposition 1

Suppose there is a symmetric equilibrium in which each player's strategy is denoted by $X(t)$. Notice that in this model, $X(t)$ is the same as the associated distribution over stopping times, $F(t)$. Any candidate symmetric equilibrium in this game shares the following properties:

(i) At time 0, none of the two players concedes with a positive probability;

(ii) $X(t)$ is continuous and strictly increasing for any t;

(iii) The game has an infinite horizon: there does not exist any t such that $X(t) = 1$.

Properties (i)-(iii) are proven by a series of subclaims:

(a) *If X^1 has a jump at $t \geq 0$, then X^2 does not jump at t.* If X^1 jumps at t, then player 2 receives a strictly higher utility by conceding an instant after t than by conceding exactly at t. This in turn implies that in a symmetric equilibrium, $X(t)$ has to be continuous.

(b) *There does not exist any t such that $X^1(t) = X^2(t) = 1$.* Suppose not and let $\hat{t} = \sup\{t : X(t) < 1\} < \infty$ be the earliest time by which the two players concede with probability one. From the continuity of $X(\cdot)$, there is no mass point at $\hat{t}$ and there exists $\tilde{t}$ such that $X(t)$ is continuous and strictly increasing on $[\tilde{t}, \hat{t}]$. Therefore, for any $t \in [\tilde{t}, \hat{t}]$,

$$0 = \int_{\tilde{t}}^{t} (v_H - cs)dX(s) + (p_0 v_L - ct)(1 - X(t)) - (p_0 v_L - c\tilde{t})(1 - X(\tilde{t})).$$

We hence obtain the following differential equation:

$$X'(t) = \frac{c}{v_H - p_0 v_L}(1 - X(t)),$$

which implies:

$$\log(1 - X(t)) - \log(1 - X(\tilde{t})) = -\frac{c}{v_H - p_0 v_L}(t - \tilde{t}).$$

For $\hat{t} < \infty$, $X(\hat{t}) < 1$ but this contradicts the definition of $\hat{t}$.

(c) *There is no interval (t_1, t_2) such that both X^1 and X^2 are constant on (t_1, t_2).* Assume the contrary and without loss of generality, let $\hat{t}$ be the supremum of t_2 for which (t_1, t_2) satisfies the above property. Fix any $t \in (t_1, \hat{t})$. Since $X(\cdot)$ is constant on $(t, \hat{t})$, each player prefers to concede at t than to concede at $\hat{t}$:

$$\int_{t}^{\hat{t}} (v_H - cs)dX(s) + (p_0 v_L - c\hat{t})(1 - X(\hat{t})) - (p_0 v_L - ct)(1 - X(t)) < 0.$$

Furthermore, from the continuity of $X(\cdot)$, the expected payoff from stopping at $\hat{t} + \epsilon$ will become arbitrarily close to the expected payoff from stopping at $\hat{t}$, as $\epsilon > 0$ converges to zero. Therefore, for $\epsilon > 0$ sufficiently small, each player prefers to concede at t than to concede at $\hat{t} + \epsilon$:

$$\int_{t}^{\hat{t}+\epsilon} (v_H - cs)dX(s) + (p_0 v_L - c(\hat{t} + \epsilon))(1 - X(\hat{t} + \epsilon)) - (p_0 v_L - ct)(1 - X(t)) < 0.$$

But this contradicts the definition of $\hat{t}$.

In models with learning, the equilibrium distribution over stopping times in any candidate symmetric equilibrium should also satisfy Properties (i)-(iii). Since the proofs are similar, we will apply these properties without reproving them in the subsequent analysis.

Properties (i)-(iii) imply that in any symmetric equilibrium, each player randomizes between conceding and staying at any time $t \geq 0$. The expected payoff for a player from stopping at t is

$$V(t;X) = \int_0^t (v_H - cs)dX(s) + (p_0 v_L - ct)(1 - X(t)).$$

A player is willing to randomize only if her payoff is constant on $[0, \infty)$. Thus, the derivative of her expected payoff with respect to t must be zero:

$$(v_H - ct)X'(t) - c(1 - X(t)) - (p_0 v_L - ct)X'(t) = 0,$$

which implies

$$x_t \triangleq \frac{X'(t)}{1 - X(t)} = \frac{c}{v_H - p_0 v_L}.$$

■

A.2 Proof of Proposition 2

The proof proceeds in three steps.

1. We construct a candidate symmetric equilibrium satisfying the equilibrium conditions.
2. We verify that the constructed equilibrium is indeed the unique symmetric equilibrium of the game.
3. We compute the expected delay in the unique symmetric equilibrium.

Step 1. Suppose there is a symmetric equilibrium in which the strategy for the high-value player has full support on an interval $[0, t')$, and the low-value player never concedes before t'. With a little abuse of notation, we use $X(\cdot)$ to denote the strategy of the high-value player.

The expected payoff for a high-value player from stopping at $t < t'$ is

$$V(t;X) = p_0 \int_0^t (v_H - cs)dX(s) + (v_L - ct)(1 - p_0 X(t)).$$

A high-value player is willing to randomize over the entire interval $[0, t')$ only if her payoff is constant on $[0, t')$. As a result,

$$p_0 X'(t)(v_H - ct) - (1 - p_0 X(t))c - p_0 X'(t)(v_L - ct) = 0, \quad \forall t < t'.$$

After simplifying the last equation, we obtain

$$\frac{p_0 X'(t)}{1 - p_0 X(t)} = \frac{c}{v_H - v_L},$$

and hence

$$p_0 X(t) = 1 - e^{-\frac{ct}{v_H - v_L}}.$$

By definition, the concession rate of the high-value player is

$$x_t \triangleq \frac{X'(t)}{1-X(t)} = \frac{c/(v_H - v_L)}{1-(1-p_0)^{\frac{ct}{v_H - v_L}}}.$$

In equilibrium, the threshold t' must have the property that the high value player concedes with probability one before t' (i.e., $X(t') = 1$). This yields

$$t' = \left(\frac{v_H - v_L}{c}\right) \log\left(\frac{1}{1-p_0}\right).$$

After t', only low-value players stay, and the equilibrium concession rate is $\frac{c}{v_H}$ to make them indifferent between conceding and staying.

Step 2. We will verify the strategies described above indeed constitute a symmetric equilibrium. First, by construction, the high-value player is indifferent between conceding and staying for all $t \leq t'$, and the low-value player is indifferent between conceding and staying for all $t > t'$.

Second, by conceding at $t \leq t'$, the low-value player's expected payoff is

$$\int_0^t (v_H - cs) d\left(1 - e^{-\frac{cs}{v_H - v_L}}\right) - cte^{-\frac{ct}{v_H - v_L}} = v_L\left(1 - e^{-\frac{ct}{v_H - v_L}}\right),$$

which is strictly positive and increasing in t. Therefore, the low-value player strictly prefers staying until time t'. Similarly, by conceding at $t > t'$, the high-value player's expected payoff is

$$\int_0^{t'} (v_H - cs) d\left(1 - e^{-\frac{cs}{v_H - v_L}}\right) + (1-p_0)\int_{t'}^t (v_H - cs) d\left(1 - e^{-\frac{c(s-t')}{v_H}}\right) + (v_L - ct)(1-p_0)e^{-\frac{c(t-t')}{v_H}},$$

which is strictly decreasing in t. Therefore, the high-value player strictly prefers conceding before time t'.

Any symmetric equilibrium must have the form described above (uniqueness), because by Lemma 1, in any symmetric equilibrium, the high-value player must concede before the low-value player.

Step 3. In the unique symmetric equilibrium, with probability p_0^2, both players concede before time t', and the expected delay is

$$\int_0^{t'} td\left[1 - \left(1 - \frac{1 - e^{-\frac{ct}{v_H - v_L}}}{p_0}\right)^2\right];$$

with probability $2p_0(1-p_0)$, one player concede before t' while the other player stays after t', and the expected delay is

$$\int_0^{t'} td\left(\frac{1 - e^{-\frac{ct}{v_H - v_L}}}{p_0}\right);$$

and with probability $(1-p_0)^2$, both players stay after t', and the expected delay is $t' + \frac{v_H}{2c}$.

Therefore, the expected delay can be written as:

$$\Omega = p_0^2 \int_0^{t'} td \left[1 - \left(1 - \frac{1 - e^{-\frac{ct}{v_H - v_L}}}{p_0}\right)^2\right] + 2p_0(1-p_0) \int_0^{t'} td \left(\frac{1 - e^{-\frac{ct}{v_H - v_L}}}{p_0}\right) + (1-p_0)^2(t' + \frac{v_H}{2c}). \tag{A.1}$$

The above expression is simplified to:

$$\begin{aligned} \Omega &= \int_0^{t'} td \left\{ 2\left[1 - e^{-\frac{ct}{v_H - v_L}}\right] - \left[1 - e^{-\frac{ct}{v_H - v_L}}\right]^2 \right\} + (1-p_0)^2(t' + \frac{v_H}{2c}) \\ &= -t' e^{-\frac{2ct'}{v_H - v_L}} + \int_0^{t'} e^{-\frac{2ct}{v_H - v_L}} dt + (1-p_0)^2(t' + \frac{v_H}{2c}). \end{aligned}$$

Plugging $e^{-\frac{2ct'}{v_H - v_L}} = (1-p_0)^2$ yields:

$$\Omega = \frac{v_H - v_L}{2c}\left(1 - (1-p_0)^2\right) + \frac{v_H}{2c}(1-p_0)^2.$$

It is verified that Ω is smaller than $\frac{v_H - p_0 v_L}{2c}$, the expected delay in the uninformed benchmark, for any $p_0 > 0$. ■

A.3 Proof of Lemma 1

The key of the proof is to show that whenever the informed player is indifferent or strictly prefers staying, the uninformed player strictly prefers staying.

Consider any symmetric equilibrium in which each player's strategy is denoted by $Z^1 = Z^2 = Z$. The associated distribution over stopping times is $F^1 = F^2 = F$. Property (ii) implies that there is no mass point in the distribution F. Hence, from equation (5), if player 1 is informed at time τ, her expected payoff from conceding at $t > \tau$ can be rewritten as

$$W^1(t;\tau, Z^2) = \int_{\tau \le s < t} (v_H - c(s-\tau))\, dF^2(s|\tau) + (v_L - c(t-\tau))\left(1 - F^2(t|\tau)\right).$$

Now assume that player 1 is uninformed at time τ, and consider the following strategy for her: she only concedes at time $t > \tau$ regardless of whether she receives a signal. The expected payoff from this strategy can be written as:

$$\begin{aligned} U^1(t;\tau, Z^2) &= \int_{\tau \le s < t} (v_H - c(s-\tau))\, dF^2(s|\tau) + (p_\tau v_L - c(t-\tau))\left(1 - F^2(t|\tau)\right) \\ &= p_\tau W^1(t;\tau, Z^2) + (1-p_\tau)\left[\int_{\tau \le s < t} (v_H - c(s-\tau))\, dF^2(s|\tau) - c(t-\tau)\left(1 - F^2(t|\tau)\right)\right]. \end{aligned}$$

If $W^1(t;\tau,Z^2) \geq v_L$, then

$$\int_{\tau \leq s < t} (v_H - c(s-\tau))\, dF^2(s|\tau) - c(t-\tau)\left(1 - F^2(t|\tau)\right) > 0,$$

which implies $U^1(t;\tau,Z^2) > p_\tau W^1(t;\tau,Z^2) \geq p_\tau v_L$. In other words, if the informed player weakly prefers to stay until t than to concede at τ, the uninformed player will strictly prefer to stay until t.

We will show that the informed players never strictly prefer staying in any candidate for a symmetric equilibrium. Suppose on the contrary there is a symmetric equilibrium with the property that there exists an interval (t_1, t_2) such that for all $t \in (t_1, t_2)$ and all $\tau < t$, we have

$$Y^1(t_2;\tau) - Y^1(t;\tau) = 0.$$

In particular, there exists $\eta > 0$ such that for all $\tau \in (t_1, t_1+\eta)$ and $t \in (\tau, t_2)$, $W^1(t;\tau,Z^2) > v_L$. Then $U^1(t;\tau,Z^2) > p_\tau v_L$ implies that if player is uninformed on the interval $(t_1, t_1+\eta)$, she strictly prefers to stay. But this violates Property (ii) since the equilibrium distribution is flat on the interval $(t_1, t_1+\eta)$.

The above analysis implies that in any candidate for a symmetric equilibrium, either the informed player is indifferent (and the uninformed player strictly prefers staying) or the informed player strictly prefers conceding. When the informed player strictly prefers conceding, the uninformed player cannot strictly prefer conceding as well because otherwise Property (ii) is violated: the equilibrium distribution over stopping times has a jump. So we are left with three possibilities as stated in the lemma. ■

A.4 Proof of Proposition 3

The proof proceeds in three steps.

1. We construct a candidate symmetric equilibrium satisfying uninformed indifference on some time interval $\mathcal{T}$.
2. We show that any symmetric equilibrium always features uninformed indifference if $\lambda p_0 \leq \frac{c}{v_H - p_0 v_L}$.
3. We verify that the constructed equilibrium is indeed the unique symmetric equilibrium of the game.

Step 1. Suppose that there is a symmetric equilibrium satisfying uninformed indifference on some time interval $\mathcal{T}$. By conceding at $t \in \mathcal{T}$, the uninformed player's expected payoff is

$$\begin{aligned} V(t) &= \int_0^t p_0 \lambda e^{-\lambda s}\left[\int_0^s (v_H - cx)dF(x) + (v_L - cs)(1-F(s))\right] ds \\ &+ \left(p_0 e^{-\lambda t} + 1 - p_0\right)\left[\int_0^t (v_H - cs)dF(s) + (p_t v_L - ct)(1-F(t))\right]. \end{aligned} \quad \text{(A.2)}$$

With density $p_0 \lambda e^{-\lambda s}$, the uninformed player becomes informed at $s \leq t$, and the associated expected payoff is $\int_0^s (v_H - cx)dF(x) + (v_L - cs)(1-F(s))$. With probability $p_0 e^{-\lambda t} + 1 - p_0$, no signal has been received, and the associated expected payoff is $\int_0^t (v_H - cs)dF(s) + (p_t v_L - ct)(1-F(t))$. Simplifying equation (A.2) yields

$$\begin{aligned} V(t) &= p_0 v_L + (v_H - p_0 v_L)\int_0^t dF(s) - c\int_0^t (1-F(s))ds \\ &- \int_0^t p_0(v_H - v_L)(1-e^{-\lambda s})dF(s) + \int_0^t p_0 c(1-e^{-\lambda s})(1-F(s))ds. \end{aligned} \quad \text{(A.3)}$$

An uninformed player is willing to randomize over the entire interval $\mathcal{T}$ only if her payoff is constant on $\mathcal{T}$. Thus, the derivative of her expected payoff with respect to t must be zero:

$$(v_H - p_0 v_L)F'(t) - c\left(1 - p_0(1-e^{-\lambda t})\right)(1-F(t)) - p_0(v_H - v_L)(1-e^{-\lambda t})F'(t) = 0,$$

which implies that

$$f_t \triangleq \frac{F'(t)}{1-F(t)} = \frac{c}{v_H - p_t v_L}.$$

Under uninformed indifference, $Y(\tau;\tau) = 1$, and from equation (4),

$$F'(t) = x_t\gamma_t + \lambda p_t \gamma_t = -\dot{\gamma}_t.$$

Hence, we obtain $\gamma_t = 1 - F(t)$, and

$$f_t = \frac{F'(t)}{1-F(t)} = x_t + \lambda p_t,$$

which implies that $x_t = f_t - \lambda p_t$.

Step 2. Next, we show that the equilibrium play is always uninformed indifference for all $t \geq 0$ if $\lambda p_0 \leq \frac{c}{v_H - p_0 v_L}$. Notice that if there exists $\epsilon > 0$ such that only the informed players concede on time interval $[0, \epsilon)$, the expected concession rate on this time interval is no more than λp_0, which is lower than $\frac{c}{v_H - p_0 v_L}$ under the assumption $\lambda p_0 \leq \frac{c}{v_H - p_0 v_L}$. This leads to a contradiction since it is also optimal for an uninformed player to concede on the time interval. Therefore, in any candidate equilibrium, uninformed indifference must occur at the beginning of the game.

Suppose by contradiction that there exists a symmetric equilibrium in which the equilibrium play is not always uninformed indifference. Denote $\tau > 0$ to be the infimum time such that the equilibrium play is not uninformed indifference. This implies that uninformed indifference occurs for $t < \tau$. In other words, a player who becomes informed at time $t < \tau$ concedes immediately at time t (i.e., $Y(z,t) = 1$ for $z \geq t$ and $t < \tau$). Then at time τ, only the uninformed players stay, and denote the posterior belief to be p_τ.

For any $t < \tau$, the law of motion for p_t satisfies $\dot{p}_t = -\lambda p_t(1-p_t)$, and hence, the time derivative of $p_t(v_H - p_t v_L)$ is

$$\frac{dp_t(v_H - p_t v_L)}{dt} = (v_H - 2p_t v_L)\dot{p}_t = -\lambda p_t(1-p_t)(v_H - 2p_t v_L),$$

which is negative because $v_H \geq 2p_0v_L$ and $p_t \leq p_0$. As a result, $p_t(v_H - p_tv_L)$ is strictly decreasing over time. For any τ, $\lambda p_\tau < \frac{c}{v_H - p_\tau v_L}$, and thus, uninformed indifference must occur in a small neighborhood of τ. But this contradicts with the definition of τ. Therefore, the equilibrium play is always uninformed indifference.

Step 3. To verify the strategies described above indeed constitute a symmetric equilibrium, we need to show that given the expected concession rate $f_t = \frac{c}{v_H - p_tv_L}$, the informed player strictly prefers conceding, and the uninformed player is indifferent between conceding and staying. The verification of the first part is trivial since $\frac{c}{v_H - p_tv_L} < \frac{c}{v_H - v_L}$. The second part is true because from equation (A.3), $V(t) = p_0v_L$ for all t. Therefore, the uninformed player is always indifferent between conceding and staying. We omit the subsequent verifications since they are similar to the one in this proof. The symmetric equilibrium is unique, because by Step 2, any symmetric equilibrium always features uninformed indifference under the assumption $\lambda p_0 \leq \frac{c}{v_H - p_0v_L}$. ■

A.5 Proof of Proposition 4

At $t = 0$, since $\gamma_0 = 1$ and $F(0) = 0$, the expected concession rate f_0 is $x_0 + \lambda p_0 Y(0; 0)$ from equation (4). f_0 is higher than λp_0 under uninformed indifference; is lower than λp_0 under informed indifference; and is equal to λp_0 under strict preferences. On the other hand, from the proof of Proposition 2, in order to make the informed players indifferent, the expected concession rate has to be $\frac{c}{v_H - v_L}$. Since $\lambda p_0 \in (\frac{c}{v_H - p_0v_L}, \frac{c}{v_H - v_L})$, it must be the case that the uninformed player strictly prefers staying, while the informed player strictly prefers conceding. As shown in the proof of Proposition 3, $p_t(v_H - p_tv_L)$ is strictly decreasing over time under the assumption $v_H \geq 2p_0v_L$. Therefore, there exists a unique T_1 such that $\lambda p_{T_1} = \frac{c}{v_H - p_{T_1}v_L}$. For $t < T_1$, $\lambda p_t \in (\frac{c}{v_H - p_tv_L}, \frac{c}{v_H - v_L})$; hence, the uninformed player strictly prefers staying, while the informed player strictly prefers conceding. For $t > T_1$, $\lambda p_t < \frac{c}{v_H - p_tv_L}$ and the equilibrium is characterized by Proposition 3. ■

A.6 Proof of Proposition 5

The proof proceeds in four steps.

1. We derive equation (8), the formula determining T.
2. We show that there is a unique $T > 0$ satisfying equation (8).
3. We prove that there exists a unique $\hat{\lambda}$ such that $\lambda p_T < \frac{c}{v_H - p_Tv_L}$ if and only if $\lambda > \hat{\lambda}$.
4. We show that $\hat{\lambda}$ is decreasing in p_0.

Step 1. Since $\lambda p_0 > \frac{c}{v_H - v_L}$, in any symmetric equilibrium, the informed players cannot concede with probability one at the beginning of the game. Otherwise, the expected concession rate $\lambda p_0 > \frac{c}{v_H - v_L}$ implies that the other player strictly prefers staying. Therefore, at the beginning of the game, the informed players have to randomize between conceding and staying; thus, the expected concession rate is $\frac{c}{v_H - v_L}$. Notice that at time t such that $\lambda p_t = \frac{c}{v_H - v_L}$, there is a positive probability of being an informed player. Property (ii) implies that the informed players cannot concede with probability one at time t because otherwise the distribution F^i will have jumps. The informed players will continue to randomize until reaching time T satisfying

$$p_0(1 - e^{-\lambda T}) = 1 - e^{-\frac{cT}{v_H - v_L}}. \tag{A.4}$$

The left-hand side of equation (A.4), $p_0(1 - e^{-\lambda T})$, is the probability that a player has received good news by time T; the right-hand side, $1-e^{-\frac{cT}{v_H - v_L}}$, is the probability that a player has conceded until time T. Therefore, the posterior probability of being an informed player reaches zero at time T, and the equilibrium play switches to either uninformed indifference or strict preferences since $\lambda p_T < \frac{c}{v_H - v_L}$.

Step 2. Let $N_1(T) = p_0(1 - e^{-\lambda T})$ and $N_2(T) = 1 - e^{-\frac{cT}{v_H - v_L}}$. Then $N_1(0) = N_2(0) = 0$ and $N_1'(T) = \lambda p_0 e^{-\lambda T}$, $N_2'(T) = \frac{c}{v_H - v_L} e^{-\frac{cT}{v_H - v_L}}$. The assumption $\lambda p_0 > \frac{c}{v_H - v_L}$ implies $\lambda > \frac{c}{v_H - v_L}$, and hence

$$\frac{N_2'}{N_1'} \propto e^{\lambda T - \frac{cT}{v_H - v_L}}$$

is strictly increasing in T, with $N_2' < N_1'$ for T close to 0. Furthermore, $N_2(T) < N_1(T)$ for T close to 0, and when $T \to \infty$, $N_2(T) = 1 > N_1(T) = p_0$. As a result, there exists at least one $T > 0$ such that $N_1(T) = N_2(T)$. Suppose there are multiple solutions satisfying equation (A.4), and let $\hat{T}$ denote the infimum among these solutions. Obviously, $N_2' > N_1'$ at $\hat{T}$, which implies that $N_2' > N_1'$ for all $t > \hat{T}$. Therefore, there is a unique $T > 0$ satisfying equation (8) because we cannot have $N_1(t) = N_2(t)$ for $t > \hat{T}$.

Step 3. Whether the equilibrium play switches to uninformed indifference or strict preferences at time T depends on the comparison of $\kappa(\lambda) = \lambda p_T(v_H - p_T v_L)$ and c. As $\lambda \to \frac{c}{p_0(v_H - v_L)}$,

$$\kappa(\lambda) \to \lambda p_0(v_H - p_0 v_L) > c.$$

As $\lambda \to \infty$, by equation (8), $T \to \bar{T}$ satisfying $p_0 = 1 - e^{-\frac{c\bar{T}}{v_H - v_L}}$, and $p_T \to 0$. As a result,

$$\lim_{\lambda \to \infty} \kappa(\lambda) = \lim_{\lambda \to \infty} \frac{\lambda p_0 e^{-\lambda T}}{p_0 e^{-\lambda T} + 1 - p_0}(v_H - p_T v_L) = 0 < c.$$

Since $\kappa(\lambda)$ is continuous in λ, there must exist at least one $\hat{\lambda} > \frac{c}{p_0(v_H - v_L)}$ such that $\kappa(\hat{\lambda}) = c$. The next step is to determine the sign of $\kappa'(\lambda)$, where

$$\kappa'(\lambda) = p_T(v_H - p_T v_L) + \lambda(v_H - 2p_T v_L)\frac{\partial p_T}{\partial \lambda}.$$

From equation (A.4), p_T can be written as

$$p_T = \frac{e^{-\frac{cT}{v_H - v_L}} - (1 - p_0)}{e^{-\frac{cT}{v_H - v_L}}}, \tag{A.5}$$

which implies that

$$\frac{\partial p_T}{\partial \lambda} = -\frac{c}{v_H - v_L}(1 - p_T)\frac{\partial T}{\partial \lambda}.$$

It is straightforward to derive from equation (A.4)

$$\frac{\partial T}{\partial \lambda} = \frac{T p_T}{\frac{c}{v_H - v_L} - \lambda p_T} > 0.$$

Hence, it is difficult to directly determine the sign of $\kappa'(\lambda)$ since $p_T(v_H - p_T v_L) > 0$ and $\lambda(v_H - 2p_T v_L)\frac{\partial p_T}{\partial \lambda} < 0$. An increase in λ causes two opposing effects in $\kappa(\lambda)$: a direct effect which increases $\kappa(\lambda)$ by increasing λ, and an indirect effect which decreases $\kappa(\lambda)$ by decreasing p_T.

However, it suffices to determine the sign of $\kappa'(\lambda)$ when $\kappa(\lambda) = \lambda p_T(v_H - p_T v_L) = c$. Plug $\kappa(\lambda) = c$ into the expression of $\kappa'(\lambda)$, and we obtain

$$\kappa'(\lambda) = \frac{c}{\lambda} - \frac{cT}{v_L}(v_H - 2p_T v_L). \tag{A.6}$$

It is still difficult to directly determine the sign of $\kappa'(\lambda)$ from equation (A.6). Equation (A.6) however implies that $\kappa'(\lambda)$ is strictly decreasing in λ under the assumption $v_H \geq 2v_L$ since T is strictly increasing in λ, and p_T is strictly decreasing in λ. This essentially guarantees the uniqueness of $\hat{\lambda}$ such that $\kappa(\lambda) < c$ if and only if $\lambda > \hat{\lambda}$. Let $\hat{\lambda}_1$ be the infimum of $\tilde{\lambda}$ satisfying $\kappa(\tilde{\lambda}) = c$, and $\kappa(\lambda) < c$ in a right neighborhood of $\tilde{\lambda}$. Suppose on the contrary that there is another $\hat{\lambda}_2 > \hat{\lambda}_1$ solving equation $\kappa(\lambda) = c$. Since $\kappa'(\hat{\lambda}_1) \leq 0$ and $\kappa'(\lambda)$ is strictly decreasing in λ when $\kappa(\lambda) = c$, we have $\kappa'(\hat{\lambda}_2) < 0$. On the other hand, $\kappa'(\hat{\lambda}_2)$ cannot be negative because $\kappa(\lambda) < c$ in a right neighborhood of $\hat{\lambda}_1$ and $\kappa(\hat{\lambda}_2) = c$. This leads to a contradiction.

Step 4. Finally, to show $\hat{\lambda}$ is decreasing in p_0, it suffices to show $\frac{\partial \lambda p_T(v_H - p_T v_L)}{\partial p_0} < 0$. From equation (A.5),

$$\frac{\partial p_T}{\partial p_0} = e^{\frac{cT}{v_H - v_L}}\left[1 - (1 - p_0)\frac{c}{v_H - v_L}\frac{\partial T}{\partial p_0}\right].$$

From equation (A.4),

$$\frac{\partial T}{\partial p_0} = \frac{1 - e^{-\lambda T}}{\frac{c}{v_H - v_L}e^{-\frac{cT}{v_H - v_L}} - p_0 \lambda e^{-\lambda T}}.$$

As a result,

$$\frac{\partial p_T}{\partial p_0} = \frac{(\frac{c}{v_H - v_L} - p_0\lambda)e^{-\lambda T}}{\frac{c}{v_H - v_L}e^{-\frac{cT}{v_H - v_L}} - p_0 \lambda e^{-\lambda T}}.$$

The denominator is strictly positive from Step 2, and the numerator is strictly negative because by assumption, $\frac{c}{v_H - v_L} - p_0\lambda < 0$. Therefore, p_T is strictly decreasing in p_0: $\frac{\partial p_T}{\partial p_0} < 0$. Then $v_H \geq 2v_L$ implies $\frac{\partial \lambda p_T(v_H - p_T v_L)}{\partial p_0} < 0$; hence, $\hat{\lambda}$ is decreasing in p_0. ∎

A.7 Proof of Proposition 6

The proof is based on the assumption that $v_H \geq 2p_0 v_L$ such that the equilibrium will stay in the uninformed indifference phase once $\lambda p_0 \leq \frac{c}{v_H - p_0 v_L}$. If the condition is not satisfied, there exists $\bar{\lambda} \leq \frac{c}{p_0(v_H - p_0 v_L)}$ such that the equilibrium will stay in the uninformed indifference phase once $\lambda \leq \bar{\lambda}$. The proof goes through similarly for $\lambda \leq \bar{\lambda}$.

The proof consists of two parts. In the first part, we show that the expected delay is increasing in λ when $\lambda p_0 \leq \frac{c}{v_H - p_0 v_L}$. In the second part, we show that the expected delay is decreasing in λ when λ approaches ∞.

Part 1. When $\lambda p_0 \leq \frac{c}{v_H - p_0 v_L}$, the expected concession rate $\frac{F'(t)}{1-F(t)}$ equals $\frac{c}{v_H - p_t v_L}$ from Proposition 3. Since $p_t = \frac{p_0 e^{-\lambda t}}{1-p_0+p_0 e^{-\lambda t}}$, the expected concession rate can be rewritten as

$$\frac{F'(t)}{1-F(t)} = \frac{c(1-p_0+p_0 e^{-\lambda t})}{v_H(1-p_0+p_0 e^{-\lambda t}) - p_0 e^{-\lambda t} v_L},$$

which implies that

$$1 - F(t) = \Psi(t,\lambda) \triangleq e^{-\frac{ct}{v_H}} \left[\frac{v_H - p_0 v_L}{(v_H - v_L) p_0 e^{-\lambda t} + (1-p_0) v_H} \right]^{-\frac{c v_L}{\lambda v_H (v_H - v_L)}}.$$

It is verified that $\frac{\partial \Psi}{\partial t} < 0$, and $\frac{\partial \Psi}{\partial \lambda}$ is proportional to

$$\frac{1}{\lambda} \log \left[\frac{v_H - p_0 v_L}{(v_H - v_L) p_0 e^{-\lambda t} + (1-p_0) v_H} \right] - \frac{(v_H - v_L) p_0 t e^{-\lambda t}}{(v_H - v_L) p_0 e^{-\lambda t} + (1-p_0) v_H}.$$

The above term is zero if $t = 0$, and the derivative with respect to t is strictly positive for all t. Therefore, we obtain $\frac{\partial \Psi}{\partial \lambda} > 0$. The expected delay can be written as:

$$\int_0^\infty t d \left(1 - \Psi(t,\lambda)^2\right) = \int_0^\infty \Psi(t,\lambda)^2 dt,$$

which converges to $\frac{v_H - p_0 v_L}{2c}$ as $\lambda \to 0$. Since $\Psi(t,\lambda)$ is increasing in λ, the expected delay must be increasing in λ as well.

Part 2. When λ approaches ∞, as shown by Proposition 5, the equilibrium consists of two phases: the informed indifference phase and the uninformed indifference phase. Let $x = p_0(1 - e^{-\lambda T})$ be the probability of concession before time T. Similar to equation (A.1), the expected delay can be written as

$$x^2 \int_0^T t d \left(1 - \left(1 - \frac{1 - e^{-\frac{ct}{v_H - v_L}}}{x} \right)^2 \right) + 2x(1-x) \int_0^T t d \left(\frac{1 - e^{-\frac{ct}{v_H - v_L}}}{x} \right)$$
$$+ (1-x)^2 \left[T + \int_0^\infty t d \left(1 - \Psi^2(t,\lambda;T)\right) \right],$$

where

$$\Psi(t,\lambda;T) = e^{-\frac{ct}{v_H}} \left[\frac{v_H - p_T v_L}{(v_H - v_L) p_T e^{-\lambda t} + (1-p_T) v_H} \right]^{-\frac{c v_L}{\lambda v_H (v_H - v_L)}}.$$

With probability x^2, both players concede before time T, and the expected delay is

$$\int_0^T td\left(1-\left(1-\frac{1-e^{-\frac{ct}{v_H-v_L}}}{x}\right)^2\right);$$

with probability $2x(1-x)$, one player concedes before T while the other player stays after T, and the expected delay is

$$\int_0^T td\left(\frac{1-e^{-\frac{ct}{v_H-v_L}}}{x}\right);$$

and with probability $(1-x)^2$, both players stay after T, and the expected delay is

$$T+\int_0^\infty td\left(1-\Psi^2(t,\lambda)\right).$$

Simplifying yields

$$\text{expected delay} = \Pi(\lambda) = \frac{v_H-v_L}{2c}(1-(1-x)^2)+(1-x)^2\int_0^\infty \Psi^2(t,\lambda;T)dt$$

and hence

$$\begin{aligned}\Pi'(\lambda) &= \frac{\partial(1-x)^2}{\partial\lambda}\left(\int_0^\infty \Psi^2(t,\lambda;T)dt-\frac{v_H-v_L}{2c}\right)\\ &+ (1-x)^2\int_0^\infty \frac{\partial\Psi^2(t,\lambda;T)}{\partial p_T}\frac{\partial p_T}{\partial\lambda}dt\\ &+ (1-x)^2\int_0^\infty \frac{\partial\Psi^2(t,\lambda;T)}{\partial\lambda}dt.\end{aligned}$$

The first term of $\Pi'(\lambda)$ is negative: a higher λ makes it more likely to receive good news, and this leads to a shorter delay because the informed players are more willing to concede; the second term of $\Pi'(\lambda)$ is positive: a higher λ leads to a lower posterior belief p_T at the beginning of the uninformed indifference phase, and this makes the uninformed players less willing to concede; and the third term of $\Pi'(\lambda)$ is positive as well: a higher λ makes the uninformed players more willing to wait in the uninformed indifference phase.

As $\lambda\to\infty$, $T\to\bar{T}$ satisfying $p_0=1-e^{-\frac{c\bar{T}}{v_H-v_L}}$, and both p_T and $\frac{\partial p_T}{\partial\lambda}$ converge to 0. Therefore, it is straightforward to verify that

$$\Pi(\lambda)\to\frac{v_H-v_L}{2c}\left(1-(1-p_0)^2\right)+(1-p_0)^2\frac{v_H}{2c},$$

which is the same as the expected delay in the full-information benchmark. Moreover,

$\lim_{\lambda\to\infty}\Pi'(\lambda)=0$, with

$$\lim_{\lambda\to\infty} e^{\lambda T}\Pi'(\lambda)=\lim_{\lambda\to\infty} e^{\lambda T}\frac{\partial(1-x)^2}{\partial\lambda}\left(\int_0^\infty \Psi^2(t,\lambda;T)dt-\frac{v_H-v_L}{2c}\right)$$
$$+\lim_{\lambda\to\infty} e^{\lambda T}(1-x)^2\int_0^\infty \frac{\partial\Psi^2(t,\lambda;T)}{\partial p_T}\frac{\partial p_T}{\partial\lambda}dt+\lim_{\lambda\to\infty} e^{\lambda T}(1-x)^2\int_0^\infty \frac{\partial\Psi^2(t,\lambda;T)}{\partial\lambda}dt.$$

By some algebra,

$$\lim_{\lambda\to\infty} e^{\lambda T}\frac{\partial(1-x)^2}{\partial\lambda}\left(\int_0^\infty \Psi^2(t,\lambda;T)dt-\frac{v_H-v_L}{2c}\right)=-2p_0(1-p_0)\frac{v_L}{2c}<0,$$

$$\lim_{\lambda\to\infty} e^{\lambda T}(1-x)^2\int_0^\infty \frac{\partial\Psi^2(t,\lambda;T)}{\partial p_T}\frac{\partial p_T}{\partial\lambda}dt=0,$$

and

$$\lim_{\lambda\to\infty} e^{\lambda T}(1-x)^2\int_0^\infty \frac{\partial\Psi^2(t,\lambda;T)}{\partial\lambda}dt=0.$$

Therefore, the expected delay is decreasing in λ when λ approaches ∞. ■

A.8 Proof of Corollary 1

The value-matching condition at p_T implies that

$$p_Tv_L=v_L+D(1-p_T)\left(\frac{1-p_T}{p_T}\right)^{\frac{c}{\lambda(v_H-v_L)}},$$

hence

$$D=-v_L\left(\frac{p_T}{1-p_T}\right)^{\frac{c}{\lambda(v_H-v_L)}}$$

and

$$V(p_0;\lambda)=v_L-v_L(1-p_0)\left(\frac{p_T}{1-p_T}\frac{1-p_0}{p_0}\right)^{\frac{c}{\lambda(v_H-v_L)}}.$$

Since $\frac{p_T}{1-p_T}\frac{1-p_0}{p_0}=e^{-\lambda T}$,

$$\left(\frac{p_T}{1-p_T}\frac{1-p_0}{p_0}\right)^{\frac{c}{\lambda(v_H-v_L)}}=e^{-\frac{cT}{v_H-v_L}}.$$

As shown by Proposition 5, T is increasing in λ. Therefore, $V(p_0;\lambda)$ is increasing in λ. ■

A.9 Proof of Proposition 7

In the bad-news model, the informed player is less willing to concede than the uninformed player; thus, the uninformed player will concede initially. Suppose there is a symmetric equilibrium in which the strategy for the uninformed player has full support on an interval $[0,T)$, and the informed player never concedes before T. Then by conceding at $t<T$, the

uninformed player's expected payoff is

$$\begin{aligned}V(t) &= \int_0^t (1-p_0)\lambda e^{-\lambda s}\left[\int_0^s (V_H - cx)dF(x) + (W(s)-cs)(1-F(s))\right]ds \\ &+ \big((1-p_0)e^{-\lambda t}+p_0\big)\left[\int_0^t (v_H - cs)dF(s) + (p_t v_L - ct)(1-F(t))\right]. \qquad \text{(A.7)}\end{aligned}$$

With density $(1-p_0)\lambda e^{-\lambda s}$, the uninformed player becomes informed at $s \le t$, and the associated expected payoff is $\int_0^s (V_H - cx)dF(x) + (W(s)-cs)(1-F(s))$. With probability $(1-p_0)e^{-\lambda t}+p_0$, no signal has been received, and the associated expected payoff is $\int_0^t (v_H - cs)dF(s) + (p_t v_L - ct)(1-F(t))$.

In equation (A.7), $W(t)$ denotes the informed player's expected payoff at time t, and satisfies

$$W(t) = \frac{\int_t^T (v_H - c(x-t))dF(x)}{1-F(t)} - c(T-t)\frac{1-F(T)}{1-F(t)}.$$

Since the uninformed player is randomizing over the interval $[0,T)$, it is also optimal for her not to concede until time T. This implies that

$$\begin{aligned}p_t v_L &= \frac{\int_t^T (v_H - c(x-t))dF(x)}{1-F(t)} + \left[\big((1-p_t)e^{-\lambda(T-t)}+p_t\big)\,p_T v_L - c(T-t)\right]\frac{1-F(T)}{1-F(t)} \\ &= W(t) + p_t v_L \frac{1-F(T)}{1-F(t)}.\end{aligned}$$

Therefore,

$$W(t) = \frac{F(T)-F(t)}{1-F(t)} p_t v_L.$$

The first order condition of equation (A.7) then implies that

$$f_t = \frac{c - \lambda(1-p_t)W(t)}{v_H - p_t v_L},$$

and hence

$$F'(t) = \frac{c(1-F(t)) - \lambda p_t(1-p_t)v_L(F(T)-F(t))}{v_H - p_t v_L}.$$

Solving the above differential equation yields

$$F(t) = \frac{\int_0^t c\Gamma(s)ds - F(T)\int_0^t \lambda v_L p_s(1-p_s)\Gamma(s)ds}{(v_H - p_t v_L)\Gamma(t)},$$

where

$$\Gamma(t) = e^{\frac{ct}{v_H}}\left[\frac{(v_H - v_L)p_0 e^{\lambda t} + (1-p_0)v_H}{v_H - p_0 v_L}\right]^{\frac{cv_L}{\lambda v_H(v_H - v_L)}}.$$

$F(T)$ is determined such that

$$F(T) = \frac{\int_0^T c\Gamma(t)dt - F(T)\int_0^T \lambda v_L p_t(1-p_t)\Gamma(t)dt}{(v_H - p_T v_L)\Gamma(T)}.$$

Finally, T is determined as follows. Notice that for $t < T$

$$\dot{\gamma}_t = -x_t\gamma_t - \lambda(1-p_t)\gamma_t$$

from equation (2) and

$$F(t) = \int_0^t x_\tau \gamma_\tau d\tau$$

from equation (4). These two equations together imply that

$$\frac{d\frac{\gamma_t}{p_0+(1-p_0)e^{-\lambda t}}}{dt} = -\frac{F'(t)}{p_0+(1-p_0)e^{-\lambda t}}.$$

Using the boundary condition $\gamma_0 = 1$, we obtain

$$p_0 - \gamma_t p_t = \int_0^t p_s F'(s)ds,$$

and thus $p_0 = \int_0^T p_s F'(s)ds$ since $\gamma_T = 0$. This leads to equation (9) by simplification. ■

References

[1] Abreu, D., Gul, F., 2000. Bargaining and reputation. Econometrica 68, 85–117.

[2] Bergin, J., MacLeod, W., 1993. Continuous time repeated games. International Economic Review 34, 21–37.

[3] Bishop, D., Cannings, C., 1978. Generalized war of attrition. Journal of Theoretical Biology 70, 85–124.

[4] Bolton, P., Harris, C., 1999. Strategic experimentation. Econometrica 67, 349–74.

[5] Bonatti, A., Hörner, J., 2011. Collaborating. American Economic Review 101, 632–63.

[6] Damiano, E., Li, H., Suen, W., 2010. Delay in strategic information aggregation. Mimeo.

[7] Damiano, E., Li, H., Suen, W., 2012. Optimal deadlines for agreements. Theoretical Economics 7, 357–93.

[8] Hendricks, K., Weiss, A., Wilson, C., 1988. The war of attrition in continuous time with complete information. International Economic Review 29, 663–80.

[9] Keller, G., Rady, S., Cripps, M., 2005. Strategic experimentation with exponential bandits. Econometrica 73, 39–68.

[10] Laraki, R., Solan, E., Vieille, N., 2005. Continuous-time games of timing. Journal of Economic Theory 120, 206–38.

[11] Nalebuff, B., Riley, J., 1985. Asymmetric equilibria in the war of attrition. Journal of Theoretical Biology 113, 517–27.

[12] Ponsati, C., Sakovics, J., 1995. The war of attrition with incomplete information. Mathematical Social Sciences 29, 239–54.

[13] Simon, L., Stinchcombe, M., 1989. Extensive form games in continuous time: Pure strategies. Econometrica 57, 1171–214.

[14] Solan, E., Vieille, N., 2001. Quitting games. Mathematics of Operations Research 26, 265–85.

[15] Weng, X., 2015. Dynamic pricing in the presence of individual learning. Journal of Economic Theory 155, 262–99.