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Star Ratings and the Incentives of Mutual Funds

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ABSTRACT

We propose a theory of reputation to explain how investors rationally respond to mutual fund star ratings. A fund's performance is determined by its information advantage, which can be acquired but decays stochastically. Investors form beliefs about whether the fund is informed based on its past performance. We refer to such beliefs as fund reputation, which determines fund flows. As performance changes continuously, equilibrium fund reputation may take discrete values only and thus can be labeled with stars. Star upgrades thus imply reputation jumps, leading to discrete increases in flows and expected performance, although stars do not provide new information.

MUTUAL FUND FLOWS ARE CLOSELY associated with fund star ratings. Analyzing Morningstar ratings, Del Guercio and Tkac (2008) show that when a fund is upgraded from the four-star group to the five-star group, on average, it observes positive abnormal flows of 53% to 61%.[1] Reuter and Zitzewitz (2015) further find that bottom five-star funds tend to receive significantly more

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[1] The abnormal flow can be interpreted as a fund's actual flow relative to its expected flow if it had maintained its prechange star rating. Newer evidence on the effect of Morningstar ratings on fund flows based on data from 2003 to 2014 is presented in Morningstar research. The observed pattern is very similar.

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flows than top four-star funds, even if their previous investment outcomes are similar. Two recent empirical studies, Evans and Sun (2018) and Ben-David et al. (2019), also document the importance of Morningstar ratings for investor decisions. We refer to the estimated effects of star ratings on mutual fund flows *star rating effects*.

However, mutual fund star ratings do not produce new information. Star ratings are coarse summaries of publicly available information. For example, the Morningstar rating is a purely quantitative, backward-looking measure of funds' past performance, it employs only five stars to summarize a fund's past investment outcomes, and since its rating mechanism is publicly available, investors can calculate star ratings based on their own knowledge of funds' previous investment outcomes. How, then, can star rating effects arise in a rational economy? Put differently, why are rational investors' investment decisions sensitive to changes in star ratings when more precise information is available?

In this paper, we study these questions by providing a natural framework for analyzing a fund's dynamic incentives to acquire private information, which has been shown to be crucial for an active mutual fund to outperform its peers.[2] Unlike a fund manager's ability, which is usually considered innate and unchanging, information can be endogenously acquired and is potentially useful for multiple periods, although it decays stochastically. Accordingly, our framework features a fund that can acquire information at a cost if it is uninformed but that may become uninformed in the next period if it is currently informed. Rational investors cannot observe whether the fund is informed or whether it has acquired information. They therefore form beliefs about whether the fund is currently informed based on its past investment outcomes and its perceived likelihood of information acquisition. We refer to such beliefs as the fund's *reputation*.

More specifically, we develop an infinite-horizon repeated game between a monopoly mutual fund and a continuum of investors. In our model, the fund's current flow is determined by the fund's current reputation and is independent of the fund's information status. Hence, an uninformed fund cannot obtain any extra flows in the current period by acquiring information. Therefore, as Berk (2005) points out, the fund's incentives to acquire information arise only from a potentially higher future reputation, which can generate larger future flows. On the one hand, when informed, the fund is more likely to realize a higher investment outcome, in which case the fund has a better reputation in the next period. On the other hand, by acquiring information in the current period, the fund may remain informed in the next period. Both effects will lead to larger future flows. Therefore, when uninformed, the fund will acquire information if and only if the incremental discounted future management fees, which we refer to as the *information premium*, can compensate for the information acquisition cost.

Our main contribution is to highlight an equilibrium phenomenon whereby the fund's reputation only takes values from a discrete set, despite the fact that

[2] For example, Phillips, Pukthuanthong, and Rau (2014) document evidence demonstrating that the source of superior fund managers' skill lies in the private information that these managers obtain.

past investment outcomes are continuously distributed. Hence, if we label these discrete values with stars, a change in the star assigned to the fund represents a discrete change in its reputation, which leads to a jump in its flows. We refer to the equilibrium property of discrete reputations as the *star rating property*.

To illustrate the star rating property, we first analytically characterize a set of equilibria in which (i) the information premium is always the same as the information acquisition cost, and hence the fund always randomizes between acquiring information and remaining uninformed; and (ii) the fund's equilibrium reputation can take one of only two possible values.

In such an equilibrium, even though fund past performance is continuously distributed, funds are endogenously partitioned into just two types: "good" and "bad." Good funds are those that investors believe are likely to have information, while bad funds are those that investors believe are less likely to have information. Funds benefit from being perceived as good because they can then attract more investors. Furthermore, there is heterogeneity among funds in each category: uninformed funds with better past performance are less likely to acquire information. This endogenous information acquisition choice leads to the same reputation for funds in the same category. Thus, a rating company that wants to accurately predict the fund's performance can group funds with the same reputation together and hence use only two stars to capture the fund's reputations.

The star rating property provides rational explanations for several empirical observations. First, the promotion of a fund to a higher rated group will naturally attract more investors. This is the star rating effect. Moreover, because in our model, funds with the same rating have the same reputation, improvements in a fund's performance and hence in its within-group rank cannot result in significant abnormal cash flows if its star rating does not change. This result is consistent with the findings documented by Del Guercio and Tkac (2008). Specifically, they show that abnormal flows are insignificantly different from zero if a fund's Morningstar percentile ranking increases but its star rating does not change.

Second, the dynamic structure of our model helps analyze star ratings' power to predict a fund's future performance. In equilibrium, higher rated funds have a higher reputation and are more likely to be informed, and so on average perform better. In other words, our theory suggests that the flow jump caused by a star rating upgrade reflects investors rationally raising their expectation of the fund's future performance. This result is consistent with empirical findings. For example, Blake and Morey (2000) show that low-rated funds have poor future performance, and Reuter and Zitzewitz (2015) show that on average the bottom five-star funds perform at least as well as the top four-star funds, and strictly outperform them in some fund classes. In addition, star ratings are imperfectly persistent: in equilibrium, high-rated funds are more likely to receive a high rating in the next period, but once they are downgraded, it is hard for them to regain a high rating.

Having illustrated the star rating property in the equilibrium with two reputation values, we argue that the star rating property is a general property of the equilibria in our model. The intuition comes from a fund's incentives. First, the probability that an uninformed fund will acquire information is strictly less than one in equilibrium, which implies that its equilibrium reputation cannot be one. This is because once investors believe that the fund is surely informed, they will not evaluate its current information status based on its current investment outcome. Thus, the benefit from generating better current investment outcomes and obtaining a higher reputation shrinks to zero. In addition, since the fund can always acquire information in the next period and delay the payment of the information acquisition cost, the benefit from staying informed in the next period is dominated by the cost of acquiring information in the current period. Therefore, the uninformed fund can profitably deviate to not acquiring information.

We show that the star rating property holds in equilibrium whenever the uninformed fund is indifferent to acquiring information.[3] Note that the information premium can be decomposed into two components: the benefit from a higher reputation and a positive discounted information premium in the next period. When the uninformed fund is always indifferent after every possible history, both the current and the discounted next-period information premiums must equal the information acquisition cost. As a consequence, the benefit from a higher reputation in the next period must be a constant as well. If the fund's equilibrium reputation set contains an open interval, then the expected benefit from obtaining a higher reputation in the next period must not change when we shift the whole distribution of investment outcomes in an open interval. But this is impossible, leading to a contradiction. By contrast, the expected benefit from obtaining a higher reputation in the next period can be constant when the fund's equilibrium reputation set includes at most countably many values. Therefore, if the fund is always indifferent in equilibrium, the equilibrium must exhibit the star rating property. When the fund sometimes shirks (not acquiring information when uninformed), the star rating property is still present when the uninformed fund is indifferent to acquiring information. The rationale is similar to that for the first case.

Our paper makes both empirical and theoretical contributions. From an empirical perspective, we provide a rational explanation for how investors respond to mutual fund star ratings, even though star ratings do not provide them with new information. Our dynamic framework can also facilitate analysis of the star ratings' ability to predict fund future performance. From a theoretical perspective, we show how a basic economic force—the trade-off between future reputation and current information acquisition cost—leads to the idea that funds benefit from being perceived as informed and work hard to preserve this status, as well as to the idea that funds with good recent performance are tempted to rest on their laurels and work less hard.

Our paper contributes to several strands of literature. Complementary to research on funds' ability, our paper contributes to the literature on funds'

[3] We show that when the information acquisition cost is sufficiently small, in equilibrium, there must be histories after which the fund is indifferent.

information acquisition.[4] García and Vanden (2009) and He (2010) show that superior information explains mutual funds' endogenous creation. Guerrieri and Kondor (2012) focus on fund managers' ability as captured by the extent to which managers are informed, which is an exogenous and permanently fixed characteristic. Kacperczyk, Van Nieuwerburgh, and Veldkamp (2014, 2016) show that mutual funds need to acquire different types of information under different scenarios to maintain a good performance record. Sockin and Zhang (2018) analyze funds' information acquisition in an equilibrium framework. Recent empirical studies demonstrate the key role of information (from various sources) in determining mutual funds' performance (Kacperczyk, Sialm, and Zheng (2005), Cohen, Frazzini, and Malloy (2008), Tang (2013), Gao and Huang (2016)). This paper contributes to this literature by developing an infinite-horizon model in which funds' information status changes both endogenously and exogenously over time, which captures the key features of information.

Our paper also contributes to recent discussions about various ratings' coarseness. Most rating systems today have finite rating notches, but the fundamentals of entities under evaluation are usually measured by continuous variables. To explain such a phenomenon, Lizzeri (1999) models an information intermediary that commits to a disclosure rule and demonstrates that the optimal rating system reveals only whether quality is above some minimal standard. Goel and Thakor (2015) study coarse credit ratings in a cheap talk model in which rating coarseness arises from the credit rating agency balancing rating accuracy and the fee charged to issuers. Harbaugh and Rasmusen (2018) argue that coarse ratings can encourage participation and thus generate more information. In this paper, the coarseness of the star rating system is an equilibrium property and arises not from information-generating/hiding purposes, but from the fact that information can be endogenously acquired and can decay stochastically. Specifically, since star ratings have a one-to-one relation with a fund's reputation, the coarseness does not provide new information nor does it lead to causes any information loss.

Finally, this paper belongs to the growing literature that views reputation as an asset that requires an investment to build and maintain.[5] The works closest to ours are Mailath and Samuelson (2001) and Board and Meyer-ter-Vehn (2013).[6] In Mailath and Samuelson (2001), an agent is either inept, that is, can

[4] Berk and Green (2004) assume that a fund's ability is innate but unknown to all players, and that a fund's expected performance is determined by its ability and size. Dasgupta and Prat (2006, 2008), Dasgupta, Prat, and Verardo (2011), and Malliaris and Yan (2015) analyze funds' reputations with respect to investor beliefs about their ability, using finite-horizon models in which a fund's ability is known only to the fund itself.

[5] There is a huge body of literature on long-lived players' reputations in game theory. For example, Kreps and Wilson (1982), Milgrom and Roberts (1982), and Fudenberg and Levine (1989) analyze an informed agent's reputation from mimicking a commitment-type player. HolmstrÖm (1999) studies a model of career concerns in which the agent's type is unknown to all players but the agent has incentives to manage other players' beliefs about her type. See Mailath and Samuelson (2014) for a survey of recent work in this area.

[6] See also Bohren (2014), Dilme (2018), Hauser (2017), and Marinovic, Skrzypacz, and Varas (2018).

exert only low effort in each period, or competent, that is, can exert high or low effort in each period. The agent's type is unobserved, and his reputation is the public belief that he is competent conditional on noisy observations of past effort. Unlike in our model, the agent's type changes randomly and exogenously, and his effort determines only current-period performance, and thus there is no star rating property in equilibrium. Board and Meyer-ter-Vehn (2013), in contrast, assume that the agent's type can change both exogenously and endogenously. However, when the agent makes decisions, she does not know her type in the current period, because there will be a random shock to her type after her decision. In their model, shirking always occurs in equilibrium. By contrast, in our model, the fund makes the information acquisition decision after the shock, and therefore when the information acquisition cost is small, an equilibrium in which the fund acquires information with positive probability always exists.

The rest of the paper is organized as follows. We describe the model in Section I and conduct preliminary analysis in Section II. In Section III, we characterize a class of equilibria in which the fund has only two possible equilibrium reputations and we derive empirical implications based on such a class of equilibria. Section IV analyzes the star rating property as a general equilibrium property and constructs other equilibria. Section V concludes. All proofs are relegated to the Appendix.

I. The Model

We consider a discrete-time infinite-horizon repeated game, where time is indexed by $t = 1, 2, \ldots$. There is a long-lived mutual fund. Also, there is a unit continuum of perfectly rational investors, each of whom considers investing in the fund in each period t.

A. Mutual Fund

The mutual fund may possess information valuable for its investment portfolio choice. We do not model the fund's optimal portfolio choice problem. Instead, we take a reduced-form approach and assume that, in each period t, the *gross* return of the fund's (optimal) investment is

$$\pi + e^{x_t},$$

where $\pi > 0$ represents the fund's fixed percentage management fee, and $e^{x_t} > 0$ is the after-fee return. The return is a random variable that follows a distribution that depends on whether the fund is informed:

$$x_t \sim \begin{cases} \mathcal{N}(1, \phi^2), & \text{if the fund is informed in period } t \\ \mathcal{N}(0, \phi^2), & \text{if the fund is uninformed in period } t. \end{cases} \tag{1}$$

Denote by $F^I(x)$ and $F^U(x)$ the corresponding distribution functions when the fund is informed and uninformed, respectively. The corresponding density functions are denoted by $f^I(x)$ and $f^U(x)$, and the likelihood ratio is denoted by $\ell(x) = f^I(x)/f^U(x) = e^{\frac{2x-1}{2\phi^2}}$, which is strictly increasing in x. We denote by $h_t = \{x_1, x_2, \ldots, x_{t-1}\}$ the history of the fund's investment outcomes up to, but not including, period t. Let $h_0 \equiv \emptyset$ denote the null history.

Whether the fund is informed is the fund's private information, and, more importantly, is endogenous. At the beginning of each period t, the uninformed fund decides whether to acquire information, while the informed fund makes no strategic choice. If the uninformed fund does not acquire information, it remains uninformed; if the uninformed fund acquires information, it becomes informed in period t. Acquiring information costs the fund $c > 0$. We denote by σ_t the probability that the uninformed fund acquires information. When $\sigma_t = 0$, the fund is said to *shirk*. We assume that investors cannot observe the fund's information acquisition decision.

Information may become *obsolete* in the sense that an informed fund in period t will become uninformed in period $t+1$ with probability $\lambda \in (0, 1)$. The assumption of $\lambda \in (0, 1)$ captures important features of information. On the one hand, $\lambda < 1$ implies that superior information has short-term persistence, and so the informed fund may still be informed in the next period. On the other hand, $\lambda > 0$ suggests that information may become obsolete stochastically.

B. Investors

Investors are uniformly distributed over the interval [0,1]. At the beginning of each period t, investor i has \$1 to invest. Observing the entire performance history of the mutual fund (h_t), the investor decides whether to invest in the fund or to take a risky outside option. An investor who invests in the fund receives an after-fee payoff e^{x_t} in the current period. We assume that if investor i decides to choose the outside option in period t, his payoff is $e^{\tilde{x}_{it}}$. Here, $\tilde{x}_{it} \sim \mathcal{N}(\bar{x}_i, \phi^2)$ is independent of the fund's investment outcome x_t and is independent across i; the mean, $\bar{x}_i$, which is known to investor i, is uniformly distributed over [0,1]. Note that the variance of the investor's outside options is the same as the variance of the fund's investment, and so the fund's investment outcome is "risk-adjusted." Hence, an investor compares the expected return from his outside option with that from investing in the fund to decide whether to delegate his money management to the fund.[7] The fraction of investors who invest in the fund is given by

$$\theta_t = \int_{i \in [0,1]} \mathbb{1}_i \, \mathrm{d}i,$$

where $\mathbb{1}_i \in \{0, 1\}$ and takes the value of one if investor i invests in the fund.

For simplicity, we assume that at the end of period t, any investor i consumes all of this end-of-period wealth, and so investors who invest in the fund will

[7] We can interpret the outside options of investors as the benefits from directly consuming their money. Such an interpretation is mathematically equivalent to our model, with investor i's benefit from directly consuming his money being $e^{\bar{x}_i + \frac{1}{2}\phi^2}$.

withdraw all of their assets from the fund. In making this assumption, we rule out the natural growth of the fund's assets under management, and hence the fund's asset under management in each period t is θ_t, which represents the fraction of investors who delegate money management tasks to the fund in period t.

C. Mutual Fund's Payoff

To capture the management fee charged by a mutual fund as a fixed percentage of the fund's assets under management, we assume that the fund's revenue in period t is $\pi\theta_t$. Therefore, given that the fund will discount its future revenues by the factor $\delta \in (0, 1)$, the fund's continuation value in any period t is

$$\sum_{\tau=t}^{\infty} \delta^{\tau-t}\big[\pi\theta_\tau - \mathbb{1}\{\text{the fund acquires information at time } \tau\}c\big], \tag{2}$$

where $\mathbb{1}\{\cdot\}$ is the indicator function.

D. Beliefs and Reputation

Investors do not observe the information status of the fund. However, they update their beliefs about the fund being informed based on the fund's information acquisition strategy and the fund's investment outcomes.

Specifically, denote by ρ_1 the common prior belief that the fund is informed at the beginning of period 1. At the beginning of each period t, investors form the period-t *prior belief* about the fund being informed at the beginning of period t,

$$\rho_t = \Pr(\text{the fund is informed at the beginning of period } t|h_t). \tag{3}$$

However, in period t, whether the fund is informed depends not only on its initial information status, but also on its information acquisition decision if it is uninformed. Based on their belief about the uninformed fund's information acquisition decision ($\hat{\sigma}_t$), investors form an *interim belief* about the fund being informed,

$$\begin{aligned} \mu_t &= \Pr(\text{the fund is informed in period } t|h_t) \\ &= \rho_t + (1-\rho_t)\hat{\sigma}_t. \end{aligned} \tag{4}$$

As we shall show in Lemma 1, the investor interim belief is the most precise indicator of the fund's expected performance conditional on its past investment outcomes, and so it determines the fund's cash flows. We therefore refer to the investor interim belief in period t as the fund's period-t *reputation*.

Once the investment outcome x_t is realized, investors update their belief about whether the fund is informed in period t, which is their *posterior belief* in period t,

$$\zeta_t = \frac{f^I(x_t)\mu_t}{f^I(x_t)\mu_t + f^U(x_t)(1-\mu_t)} = \frac{\ell(x_t)\mu_t}{\ell(x_t)\mu_t + 1 - \mu_t}. \tag{5}$$

Because the information will be obsolete with probability λ, investors will discount the period-t posterior belief by such a probability to form the period-$(t+1)$ prior belief,

$$\rho_{t+1} = (1-\lambda)\zeta_t. \tag{6}$$

Thanks to the monotonicity of the likelihood ratio function $\ell(\cdot)$, ζ_t is strictly increasing in x_t for all μ_t. In other words, a good investment outcome always increases the current-period investor posterior belief and hence the next-period investor prior belief, regardless of the fund's strategy and its previous performance.

E. Timing

Figure 1 summarizes the timing in each period t.

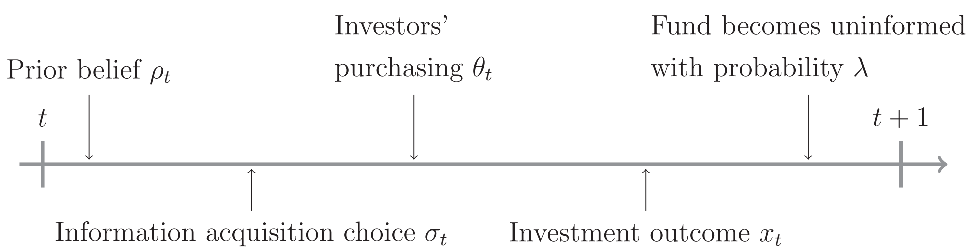


Figure 1. Timing of the game.

F. Equilibrium

We solve for a *monotone Markov perfect equilibrium*, where the state variable is the investor prior belief ρ_t and the fund's reputation μ_t is weakly increasing in ρ_t. The unbounded likelihood ratio property of the random variable x_t implies that the investor posterior belief ζ_t belongs to [0,1], and so equation (6) implies that $\rho_t \in [0, 1-\lambda]$ in any period $t \geq 2$. We further assume that the common prior $\rho_1 \in [0, 1-\lambda]$. The assumption is made for simplification since the interval $(1-\lambda, 1]$ cannot be reached in any continuation play. Hence, without loss of generality, we restrict the Markov state space to $[0, 1-\lambda]$. Unless otherwise specified, in the rest of the paper, we drop the subscript t, denote by ρ (and μ) the current-period prior belief (and reputation), and denote by ρ' (and μ') the next-period prior belief (and reputation).

Formally, a *monotone Markov perfect equilibrium* is specified by (i) the information acquisition policy of the fund $\sigma : [0, 1-\lambda] \to [0, 1]$, (ii) the fraction of investors who invest in the fund $\theta : [0, 1-\lambda] \to [0, 1]$, and (iii) the belief updating function $\rho' : [0, 1-\lambda] \times \mathbb{R} \to [0, 1-\lambda]$ such that

(1) given $\theta(\cdot)$, $\rho'(\cdot)$, and ρ, $\sigma(\rho)$ maximizes the fund's expected continuation payoff (2),

(2) given $\sigma(\cdot)$, $\rho'(\cdot)$, and ρ, the fund's reputation is

$$\mu(\rho) = \rho + (1-\rho)\sigma(\rho), \tag{7}$$

where μ is a left-continuous, weakly increasing function of ρ for all $\rho \in [0, 1-\lambda]$, and the fraction of investors who find it optimal to invest in the fund is

$$\theta(\rho) = \int_{i\in\mathcal{D}} \mathbb{1}_i di, \tag{8}$$

where

$$\mathcal{D} = \{i \in [0,1] : \mu(\rho)\mathbb{E}(e^x|F^I) + (1-\mu(\rho))\mathbb{E}(e^x|F^U) \geq \mathbb{E}(e^{\tilde{x}_i}|\bar{x}_i)\},$$

and

(3) given $\sigma(\cdot)$, $\theta(\cdot)$, ρ, and the current-period investment performance x, the state variable in the next period is

$$\rho'(\rho, x) = \frac{(1-\lambda)\ell(x)\mu(\rho)}{\ell(x)\mu(\rho) + 1 - \mu(\rho)}. \tag{9}$$

In the definition of our solution concept, the set $\mathcal{D}$ consists of the investors whose outside options generate lower expected returns than the fund. The belief updating function in equation (9) is pinned down by equations (4) to (7). The only unconventional requirement is that, in equation (7), the fund's reputation is weakly increasing with the investor prior belief. This restriction is plausible for purposes of modeling reputations as assets and studying the reputation dynamics of mutual funds. It ensures that the fund values a good reputation regardless of its information status (see Proposition 1).[8] On the one hand, it is natural for investors to believe that a fund that is more likely to be informed at the beginning of the period is also more likely to be informed when making an investment (before any new signals about whether the fund is informed). On the other hand, when reputation is increasing with the investor prior belief, a better current investment outcome ensures a higher reputation and hence a larger fund flow in the next period. These implications are consistent with a basic empirical regularity in the literature, namely, that investors chase past performance.

II. Preliminary Analysis

In this section, we first show that, in equilibrium, investors follow a cutoff rule based on the fund's reputation in each period. We then derive the fund's value functions and define the fund's information premium. The latter is

[8] Such a restriction is also crucial to establish the star rating property, as we explain in detail in Section IV. In the Internet Appendix, however, we numerically construct a nonmonotone equilibrium, which is available in the online version of the article on *The Journal of Finance* website.

crucial to understanding the fund's information acquisition incentives. We then argue that the fund will never strictly prefer to acquire information in equilibrium, which not only simplifies further equilibrium analysis but also has important implications for the characterization of equilibrium in Section IV. Finally, we show that the fund's payoff is increasing in its reputation thanks to the monotonicity condition of the equilibrium.

A. Cash Flows

In each period, an investor needs to choose between buying the fund's shares and his outside option. Given the fund's strategy and the current state variable ρ, when investors make decisions, they believe that the fund is informed with probability μ, as defined in equation (7). Therefore, investor i will choose the fund if and only if the expected after-fee return of the fund exceeds his outside option,

$$\mu\mathbb{E}(e^x|F^I) + (1-\mu)\mathbb{E}(e^x|F^U) \geq \mathbb{E}(e^{\tilde{x}_i}|\bar{x}_i). \tag{10}$$

Since $\bar{x}_i$ is uniformly distributed over [0,1], equation (10) directly implies Lemma 1 below, which characterizes the fraction of investors who buy the fund's shares in equilibrium.

LEMMA 1: *In equilibrium, the fraction of investors who buy the fund's shares depends on the state variable ρ only through the fund's reputation μ, that is,*

$$\theta(\rho) = \ln\left[\mu(\rho)e + (1-\mu(\rho))\right]. \tag{11}$$

Equation (11) shows that the fund's cash flows are strictly increasing in its reputation. Therefore, the fund's reputation is indeed an important asset that can generate cash flows.

B. Value Function and Information Premium

We now analyze the fund's value functions and define the information premium. Given the investor prior belief $\rho \in [0, 1-\lambda]$, the fund's equilibrium continuation value is

$$W_I(\rho) = \pi\theta(\rho) + \delta\int_{-\infty}^{\infty}\left[(1-\lambda)W_I(\rho') + \lambda\max\{W_U(\rho'), W_I(\rho') - c\}\right]\mathrm{d}F^I(x) \tag{12}$$

if it is informed and

$$W_U(\rho) = \pi\theta(\rho) + \delta\int_{-\infty}^{\infty}\max\{W_U(\rho'), W_I(\rho') - c\}\mathrm{d}F^U(x) \tag{13}$$

if it is uninformed, where $\theta(\rho)$ is given by investors' equilibrium strategy, and the value of ρ' is determined by ρ and the realization of x (by equation (9)).

The economic interpretations of equation (12) are as follows. First, the fund's current-period cash flow $\theta(\rho)$ is determined by investors' prior belief about the state and their belief about the fund's information acquisition decision. Thus, importantly, the fund's current-period cash flow is independent of its current information status and its information acquisition decision. Second, because the fund is informed, its investment outcome will be drawn from the distribution with the cumulative distribution function (cdf) $F^I(x)$. Given each realized investment outcome x, the next-period investor prior belief ρ' can be calculated by equation (9). Third, with probability $1-\lambda$, the fund will stay informed, and so its continuation value in the next period will be $W_I(\rho')$. But with probability λ, the fund will become uninformed and in such a case, it will need to decide whether to acquire information in the next period, and so, its next-period continuation value will be $\max\{W_U(\rho'), W_I(\rho') - c\}$. The economic interpretation of equation (13) is similar.

We refer to $W_I(\rho) - W_U(\rho)$ as the fund's *information premium* at ρ, which corresponds to the benefit from being informed. Then the fund's optimal information acquisition strategy is

$$\sigma(\rho)\begin{cases} = 0, & \text{if} \quad W_I(\rho) - W_U(\rho) < c \\ \in [0, 1], & \text{if} \quad W_I(\rho) - W_U(\rho) = c \\ = 1, & \text{if} \quad W_I(\rho) - W_U(\rho) > c. \end{cases} \tag{14}$$

That is, an uninformed fund acquires information if and only if the benefit from doing so is not less than the cost. Because the information premium is bounded, when c is large the fund always shirks, that is, $\sigma(\rho) = 0$ for all ρ. To avoid such a trivial case, in the rest of the paper, we assume that c is sufficiently small.

Equations (12) to (14) demonstrate the importance of the assumption of $\lambda < 1$. Specifically, it follows from equation (9) that if $\lambda = 1$, $\rho'(x) = 0$ regardless of the realization of x, that is, it is common knowledge that the fund will surely be uninformed at the beginning of the next period. Consequently, the right-hand side of equation (12) equals that of equation (13) (since $\rho'(x) = 0$ for all x). As a result, the fund's information premium is zero, and hence it follows from equation (14) that the fund will not acquire information. The underlying intuition is simple. Because in our model, the mutual fund does not charge a performance fee, acquiring information can only be rewarded in the future due to the likelihood that the fund is informed. When $\lambda = 1$, acquiring information today cannot affect the fund's future information status, and so the fund has no incentive to do so. We present more comparative statics about λ in the Internet Appendix.

C. Information Acquisition Strategy

We now discuss the incentive of an uninformed fund to acquire information in equilibrium. The benefit of acquiring information is quantified by the value of the information premium. Since an informed fund is more likely to have a better investment outcome, its reputation is more likely to be high in subsequent periods, which, in turn, attracts more cash flows.

However, when the fund is believed to acquire information at some prior belief, the fund has the incentive to deviate. To see this, suppose that there is an equilibrium in which the uninformed fund acquires information with probability 1 when the investor prior belief is ρ, that is, $\sigma(\rho) = 1$. In such a case, the fund's reputation will be 1. Investors will not update their beliefs about whether the fund is informed based on its investment outcome. Hence, the next-period investor prior belief will be independent of the fund's current investment outcome, undermining the fund's information premium.

Although the fund can still benefit from information acquisition because it can remain informed with positive probability, such a benefit is dominated by the information acquisition cost. This can be seen by considering a deviation in which the fund delays acquiring information in the next period. By such a deviation, the fund will also be informed in the next period, but will pay the information acquisition cost later. Given the fund's discount factor $\delta < 1$, such a deviation is profitable. Lemma 2 below summarizes the arguments outlined above.[9]

LEMMA 2: *In equilibrium, the fund never strictly prefers to acquire information. Formally, in equilibrium,* $\sigma(\rho) < 1$ *for any* $\rho \in [0, 1 - \lambda]$.

Lemma 2 greatly simplifies our analysis. Generally, $W_U(\cdot)$ is the continuation value obtained by assuming that the fund chooses to remain uninformed in the current period. To incorporate the information acquisition decision, the value function of an uninformed fund is given by $\max\{W_U(\rho), W_I(\rho) - c\}$. Lemma 2 states that in equilibrium, the uninformed fund will never acquire information with probability 1, which, in turn, implies that, for all $\rho \in [0, 1 - \lambda]$, we have $W_U(\rho) \geq W_I(\rho) - c$. Therefore, $\max\{W_U(\rho), W_I(\rho) - c\} = W_U(\rho)$, and $W_U(\cdot)$ is the value function of the uninformed fund.

D. The Value of Good Reputation

We now argue that in a *monotone* Markov perfect equilibrium, a fund always prefers a better reputation. As the fund's flow payoff is determined by the state variable ρ through its reputation μ only, we find it more convenient to express the fund's continuation value as a function of its reputation. We denote by $V_I(\mu)$ and $V_U(\mu)$ the fund's value function of its reputation when it is informed and uninformed, respectively. It then follows that $W_I(\rho) = V_I(\mu(\rho))$ and $W_U(\rho) = \sigma(\rho)[V_I(\mu(\rho)) - c] + (1 - \sigma(\rho))V_U(\mu(\rho))$. From Lemma 2, in equilibrium, an uninformed fund never finds it strictly optimal

[9] One key assumption for Lemma 2 is the certainty of being informed once the fund acquires information. Indeed, if the fund becomes informed with probability $q < 1$ when it acquires information, there may be an equilibrium in which the fund acquires information for sure when the investor prior belief is above some threshold (since the fund's equilibrium reputation must be weakly increasing in the investor prior belief).

to acquire information, and so $V_I(\mu(\rho)) - c \le V_U(\mu(\rho))$ for all ρ, and $\sigma(\rho) = 0$ if $V_I(\mu(\rho)) - c < V_U(\mu(\rho))$. Hence, $W_U(\rho) = V_U(\mu(\rho))$. Therefore, we have

$$V_I(\mu) = \pi\hat{\theta}(\mu) + \delta \int_{-\infty}^{\infty} \left[(1-\lambda)V_I(\mu') + \lambda V_U(\mu')\right]\mathrm{d}F^I(x) \tag{15}$$

and

$$V_U(\mu) = \pi\hat{\theta}(\mu) + \delta \int_{-\infty}^{\infty} V_U(\mu')\mathrm{d}F^U(x), \tag{16}$$

where the current-period cash flow $\hat{\theta}(\mu(\rho)) = \theta(\rho)$ is determined by equation (11), and the next-period reputation μ' is a function of the current reputation μ and the current investment outcome x,

$$\mu'(\mu, x) = \rho' + (1-\rho')\sigma(\rho'), \tag{17}$$

where $\rho' = \frac{(1-\lambda)\ell(x)\mu}{\ell(x)\mu+1-\mu}$. Notice that because μ' is weakly increasing in ρ' in a monotone equilibrium, μ' is weakly increasing in x for all μ and also weakly increasing in μ for all x. That is, a higher investment outcome always increases the fund's next-period reputation, and a fund with a higher reputation is more likely to end up with a higher reputation in the future.

PROPOSITION 1: *In equilibrium,*

(1) the fund's continuation value $V_j(\mu)$ is strictly increasing in μ for $j = U, I$, and

(2) for any $\mu \in (0, 1)$, $0 \le V_I(\mu) - V_U(\mu) \le c$.

The first part of Proposition 1 shows that the fund's continuation value is strictly increasing in its reputation regardless of its information status. This is intuitive. By equation (11), the fund's current cash flow is strictly increasing in its reputation. Also, a fund with a higher reputation can always replicate the equilibrium strategy of a fund with the same information status but a lower reputation in the continuation plays and receives strictly larger cash flows. Finally, since $\mu'(\mu, x)$ is weakly increasing in μ for all x, a fund will be rewarded for its good reputation not only in the current period but also in the future. Notice that Proposition 1 depends crucially on the equilibrium restriction that the fund's reputation is weakly increasing in investor prior beliefs. Otherwise, when $\mu'(\rho')$ is decreasing at some ρ', the fund may be better off when it has a worse investment outcome (which leads to a lower investor prior belief).

With the fund's value functions of its reputation at hand, we can now redefine the information premium at the reputation level μ as $V_I(\mu) - V_U(\mu)$. Obviously, since the reputation μ is a function of the investor prior belief ρ, we have $V_I(\mu(\rho)) - V_U(\mu(\rho)) = W_I(\rho) - W_U(\rho)$. Hence, the second part of Proposition 1 shows that the equilibrium information premium is always bounded between 0 and c.

III. Two-Star Equilibria

In this section, we construct a set of equilibria in which the fund's equilibrium reputation set includes only two values. The analysis of such an equilibrium illustrates several interesting equilibrium properties. Importantly, while the investor prior belief is continuously distributed, the fund's reputation, as a function of investor prior belief, takes finitely many values in equilibrium. We refer to this feature as the star rating property. Formally, we say that an equilibrium exhibits the *star rating property* if there is an open interval $\mathcal{V} \subset [0, 1-\lambda]$ of investor prior beliefs such that the fund's corresponding reputation can take at most countably many values. Furthermore, when the star rating property holds globally, that is, $\mathcal{V} = [0, 1-\lambda]$, we say that the equilibrium exhibits the *strong star rating property*. We discuss the empirical implications of the equilibrium with two stars in Section III.C.

A. A Star rating Equilibrium with Two Stars

In this subsection, we construct an equilibrium in which the fund's reputation can take only two values. Therefore, we can label the fund's reputations with stars and call such an equilibrium a *star rating* equilibrium (*SRE*) with two stars. By constructing such an equilibrium, we also prove the existence of an equilibrium of the model.

PROPOSITION 2: *There exists $\bar{c} > 0$ such that for any $c < \bar{c}$, the model has an SRE with two stars. The SRE is fully characterized by $(\hat{\rho}, \mu_1^*, \mu_2^*)$ as follows:*

(1) if the fund is uninformed, its information acquisition strategy is

$$\sigma(\rho) = \begin{cases} \frac{\mu_1^* - \rho}{1-\rho} & \text{if } \rho \in (0, \hat{\rho}] \\ \frac{\mu_2^* - \rho}{1-\rho} & \text{if } \rho \in (\hat{\rho}, 1-\lambda), \end{cases} \tag{18}$$

and

(2) the fund cash flow θ^* specified in equation (11) is determined by μ_1^* and μ_2^*.

Equations (4) and (18) imply that the equilibrium characterized in Proposition 2 features only two equilibrium reputations. Specifically, the funds with investor prior beliefs less than or equal to $\hat{\rho}$ have the same reputation μ_1^* and hence can be grouped together; we call them *one-star* funds. In contrast, the funds with investor prior beliefs strictly greater than $\hat{\rho}$ have the same reputation μ_2^*, and therefore, can be grouped together and called *two-star* funds. Thus, the equilibrium reputation is a step function of investor prior belief.

Since the fund's reputation can take only two possible values, given the fund's information status, its value function can have only two values too. Therefore, to keep the notation simple, we rewrite the fund's value function as V_j^K, which denotes the fund's continuation value when it has information status $K \in \{I, U\}$ and reputation μ_j^* ($j = 1, 2$). Note that the fund's information

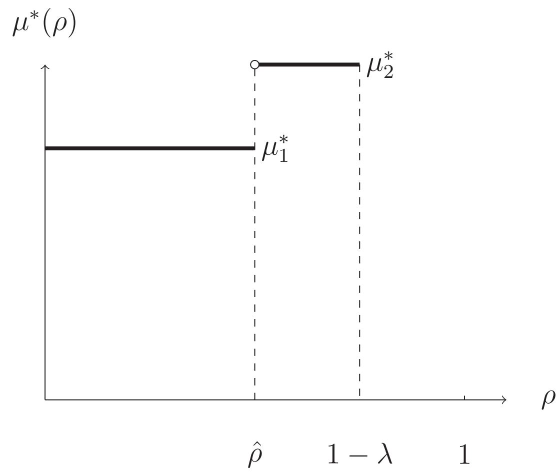


Figure 2. The fund's reputation, μ^*, as a function of investor prior belief, ρ.

acquisition strategy characterized in equation (18) shows that the fund is playing a mixed strategy for almost all investor prior beliefs, and so the fund must be indifferent between acquiring information and remaining uninformed, given the cash flows specified in equation (11). Therefore,

$$c = V_1^I - V_1^U = V_2^I - V_2^U, \tag{19}$$

where the law of motion of μ is determined solely by $\hat{\rho}$. We show that when c is sufficiently small, there is a solution $(\hat{\rho}, \mu_1^*, \mu_2^*)$, which establishes the existence of an equilibrium.

Figure 2 illustrates the equilibrium reputation as a function of investor prior belief. It suggests that in an SRE with two stars, a two-star fund has a higher reputation than a one-star fund, that is, $\mu_2^* > \mu_1^*$. This stems from the reputation's role in incentivizing the fund to acquire information. Otherwise, if $\mu_2^* = \mu_1^*$, the fund will strictly prefer not to acquire information, because it will always get a constant cash flow in each period regardless of its investment outcome.

In the equilibrium characterized in Proposition 2, the fund's information acquisition probability is not monotonic over the whole space of the investor prior beliefs. Specifically, as investor prior beliefs increase, if the fund's star rating does not change, the fund is less likely to acquire information. However, a bottom two-star fund is much more likely to acquire information than a top one-star fund. Figure 3 illustrates the fund's information acquisition strategy.

Figure 3 also illustrates the importance of the assumption that $\lambda > 0$. Because equilibrium reputation must be weakly increasing in investor prior belief, the reputation level of a two-star fund, μ_2^*, must be at least $1 - \lambda$. Thus, if $\lambda = 0$, μ_2^* must be equal to 1, implying that all two-star funds must acquire information with probability 1 in equilibrium. This violates the conclusion

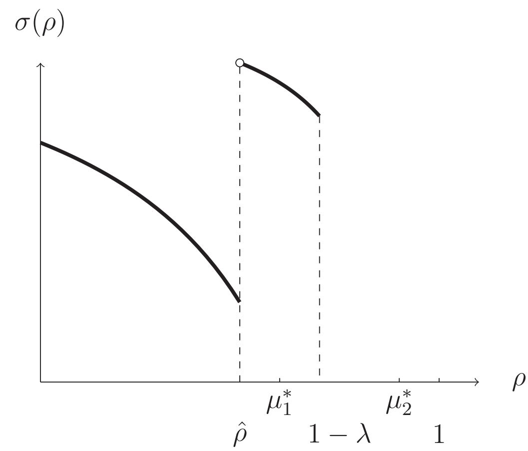


Figure 3. The fund's information acquisition strategy, σ, as a function of investor prior belief, ρ.

that we draw in Lemma 2. Hence, if $\lambda = 0$, there does not exist such a μ_2^*, and thus an SRE with two stars does not exist either.

B. Equilibrium Properties

There are some interesting properties of the equilibrium characterized in Proposition 2.

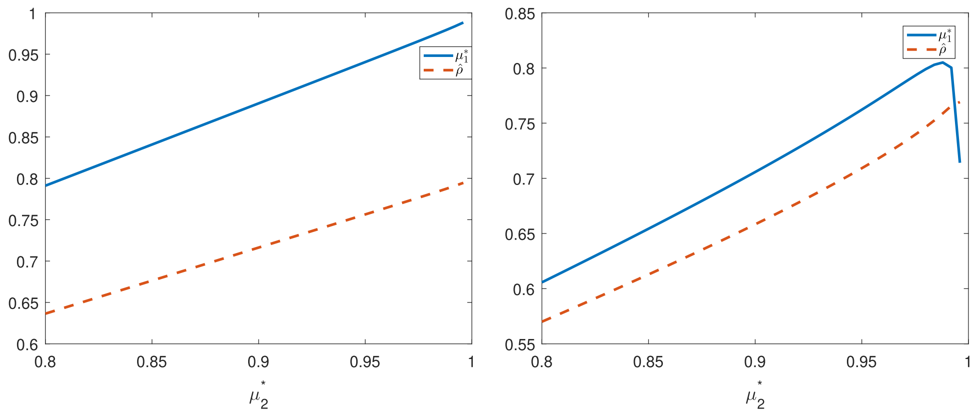
B.1. Equilibrium Multiplicity

We find that there are multiple SREs with two stars. This is not surprising due to the self-fulfilling nature of the model (the fund best responds to investors' beliefs about its information acquisition behavior). Technically, there are three unknowns in equation (19), and so the solution to it is generally not unique. We formalize this in Corollary 1.

COROLLARY 1: *For any $c < \bar{c}$, there is a continuum of SREs with two stars.*

Figure 4 depicts how the set of equilibria changes with the information acquisition cost c. In the left panel, c is very small, and hence for all $\mu_2^* \in [1 - \lambda, 1)$, there exists $\mu_1^* < \mu_2^*$ such that μ_1^* and μ_2^* are reputations in an equilibrium. In addition, different equilibria have different cutoffs of star ratings ($\hat{\rho}$), and in an equilibrium with a higher cutoff, the fund has higher reputations for both one-star and two-star ratings.

In the right panel, c is relatively large. In this case, the reputations of both one-star funds and two-star funds also increase as the cutoff $\hat{\rho}$ goes up. However, the reputation μ_2^* cannot be arbitrarily close to 1, because μ_1^* drops below $\hat{\rho}$ when μ_2^* becomes very large. Thus, the highest attainable μ_2^* in an SRE is achieved when $\mu_1^* = \hat{\rho}$.



Panel A. $c = 0.01$ Panel B. $c = 0.3$

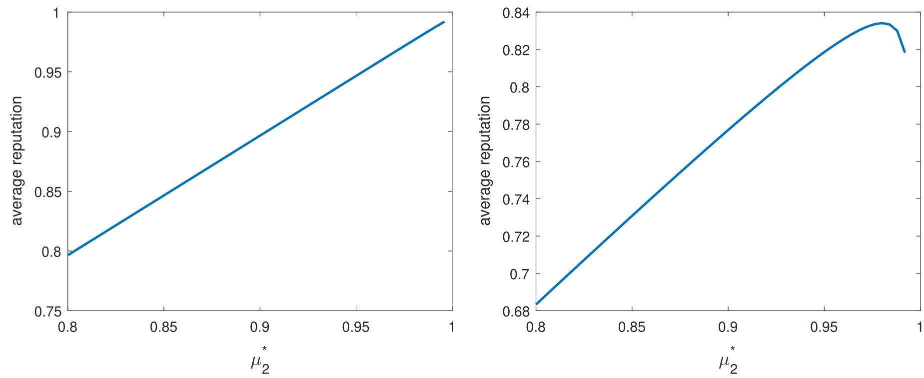
Figure 4. Multiple SREs. The panels plot the existence of multiple SREs and how the set of SREs changes with the information acquisition cost, c. The other parameters are $\delta = 0.9$, $\lambda = 0.2$, and $\phi = 1$. (Color figure can be viewed at wileyonlinelibrary.com)

B.2. Average Reputation

Because there exist multiple SREs with two stars, it would be interesting to analyze welfare across equilibria. We measure welfare using the average reputation of the fund, which is given as the sum of all agents' unconditional (on ratings) payoffs in the model. First, in the SRE, an uninformed fund is always indifferent between acquiring information and remaining uninformed, so its payoff is the same as when it chooses to remain uninformed. Note that such a value is equal to the discounted management fees paid by investors, the money transfers from investors to the fund. Hence, welfare is just the fund's average productivity. Conditional on the star rating, the fund's productivity can be measured as the objective probability that it is informed, which is equal to the fund's reputation, so the fund's conditional productivity (on the star rating) is the fund's conditional reputation. Hence, unconditional (on star ratings) welfare should be measured by the fund's average reputation.[10]

Since an equilibrium features a unique corresponding reputation μ_2^*, we analyze how the fund's average reputation changes with μ_2^*. Figure 5 plots average reputation as a function of the reputation of a two-star fund for different information acquisition costs. The left panel of the figure shows that when the information acquisition cost is small, the average reputation can be arbitrarily close to 1 in equilibrium. This is because when the information acquisition cost is small, there exists an equilibrium with two stars in which both μ_1^* and μ_2^* are arbitrarily close to 1, as illustrated in Figure 4, Panel A. However, from

[10] Notice that we focus on the case in which c is so small that it is possible to incentivize the fund to acquire information. As a consequence, it must be socially efficient for an uninformed fund to acquire information.



Panel A. $c = 0.01$ Panel B. $c = 0.3$

Figure 5. Average reputation. The panels plot the average reputation across equilibria with two stars, for different information acquisition costs c. (Color figure can be viewed at wileyonlinelibrary.com)

Lemma 2, it follows that $\mu_2^* < 1$, so there is no maximum average reputation across equilibria with two stars when the information acquisition cost is small.

More interestingly, the right panel of Figure 5 shows that the equilibrium in which the average reputation is maximized differs from the equilibrium in which the two-star fund's reputation (μ_2^*) is maximized. This is because if we want to realize a higher μ_2^*, we may have a larger fraction of one-star funds with a lower reputation. When such an effect dominates, an equilibrium with a higher μ_2^* will have a lower average reputation (or a lower average productivity).

B.3. Implementation by a Simple Star Rating System

The two-star equilibrium is easy to implement in practice. Suppose that a rating company commits to the following rating system:

$$\iota = \begin{cases} 1, & \text{if } \rho \leq \hat{\rho} \\ 2, & \text{if } \rho > \hat{\rho}. \end{cases} \tag{20}$$

While the investor prior belief (ρ) is unobservable, in our model, it is a function of the fund's entire performance history. Therefore, in equilibrium, the star rating system (equation (20)) is equivalent to a performance-based *simple rating rule*, which is specified below. Let x_j^* be the investment outcome such that a j-star fund with x_j^* will be perceived by investors as informed with probability $\hat{\rho}$ at the beginning of the next period, that is,

$$\hat{\rho} = \frac{(1-\lambda)\ell(x_j^*)\mu_j^*}{\ell(x_j^*)\mu_j^* + 1 - \mu_j^*},$$

where the right-hand side is obtained by substituting μ_j^* into the equilibrium belief-updating function (9). Because $\ell(\cdot)$ is strictly increasing, a j-star fund will be assigned a two-star rating in the next period if and only if its current investment outcome $x > x_j^*$.

On the one hand, the inputs of such a simple rating rule, that is, funds' most recent star ratings and their current investment outcomes, are both observable and verifiable in contrast to the (subjective) belief. On the other hand, a fund's most recent star rating summarizes its previous performance. Therefore, the stars generated by the simple rating rule provide investors with sufficient quantitative evaluations of funds' past performance, making it unnecessary to track the entire performance history of the fund to "calculate" its reputation.

C. Empirical Implications

We now discuss several empirical implications of the star rating property and relate them to the documented empirical observations. To discuss the empirical implications, we also assume that there is a rating agency that commits to assigning ratings to mutual funds according to the rating rule specified in equation (20). While the discussions in this section are based on the SRE constructed in Proposition 2 for simplicity, we emphasize that these empirical implications hold in all equilibria satisfying the star rating property.

C.1. Star Rating Effects on Funds' Cash Flows

Because a j-star fund has the reputation μ_j^*, it follows from Lemma 1 that, in equilibrium, a j-star fund receives flows $\theta_j^* = \ln[\mu_j^* e + (1 - \mu_j^*)]$. Corollary 2 shows that the fund receives positive flows when it is upgraded from one star to two stars.

COROLLARY 2: *In an SRE with two stars, when the fund is upgraded from one star to two stars, it will receive net flows*

$$\theta_2^* - \theta_1^* > 0. \tag{21}$$

Corollary 2 shows the famous star power phenomenon, which is documented in Del Guercio and Tkac (2008). In general, Del Guercio and Tkac (2008) find a consistent flow response across rating-change categories, namely, positive for rating upgrades and negative for rating downgrades.[11] Such empirical observations are consistent with the prediction in Corollary 2, and so our result implies that the star power phenomenon can arise in a fully rational model.

Another interesting property of an SRE is the discontinuity of the fund's reputation in the investor prior belief. It follows from equation (18) and

[11] In our model, we abstract from many other factors that may also affect funds' cash flows and focus on the effects of star ratings. This is consistent with the empirical method employed by Del Guercio and Tkac (2008). Specifically, they first consider a benchmark model that filters out the effects of other performance measures on funds' cash flows. They then define a fund's "abnormal flows" as the difference between the fund's actual flows and its benchmark cash flows.

$\mu_2^* > \mu_1^*$ that the fund's reputation is discontinuous at $\hat{\rho}$. Therefore, for any $\epsilon > 0$, if two funds, Fund A and Fund B, have investor prior beliefs ρ_A and ρ_B, respectively, and $\hat{\rho} - \epsilon < \rho_A \leq \hat{\rho} < \rho_B < \hat{\rho} + \epsilon$, then Fund A will receive a one-star rating, while Fund B will get a two-star rating. As a consequence, $\mu_A = \mu_1^* < \mu_2^* = \mu_B$, which implies that Fund A is strictly smaller than Fund B. That is, although Fund A and Fund B are considered almost the same in terms of their information status at the beginning of the period, Fund B is more popular than Fund A among investors.

This implication is consistent with one of the key observations documented in Reuter and Zitzewitz (2015). The authors show that a small change in a fund's investment outcome may lead to significant cash flows. In particular, before assigning star ratings, Morningstar first ranks all funds to calculate their percentile rankings. It then uses predetermined cutoffs in the distribution of the percentile rankings to assign star ratings.[12] Therefore, cross-sectionally, funds with percentile rankings around the cutoffs will be assigned different stars and thus have different size. This is exactly what Reuter and Zitzewitz (2015) document: mutual funds just above the threshold for a Morningstar rating receive incremental net flows that are about 2.5% of the assets under management.

In contrast to the significant cash flows in the event of a star rating change, a fund with a star rating that remains the same will not receive significant abnormal cash flows, even if its investment outcome improves. In an SRE, when a fund is ranked higher within a star group (i.e., the investor prior belief is higher), its size does not change and so its net cash flow is zero. This prediction is summarized in Corollary 3.

COROLLARY 3: *In an SRE, if a fund gets the same rating in periods t and t+1 according to the rating system (20), then the fund's net cash flow in period t+1 is zero.*

This prediction is also consistent with the empirical evidence documented by Del Guercio and Tkac (2008), who isolate the star rating's effect from flow responses to other performance measure changes. They first show that funds' abnormal flows are insignificantly different from zero in no-rating-change periods. They then show that, overwhelmingly, when funds' percentile rankings change but their star ratings do not change, their abnormal flows continue to be insignificantly different from zero.

C.2. Star Rating Predictive Power

In an SRE with two stars, $\mu_2^* > \mu_1^*$. Therefore, a two-star fund is more likely to be informed than a one-star fund. Because a fund's information acquisition strategy is consistent with investors' beliefs, we expect a two-star fund to perform better than a one-star fund. This intuition is confirmed by Corollary 4.

[12] After calculating funds' percentile rankings, Morningstar assigns five stars to funds in the top 10%, four stars to funds in the next 22.5%, three stars to funds in the middle 35%, two stars to funds in the next 22.5%, and one star to funds in the bottom 10%.

COROLLARY 4: *In an SRE with two stars, the expected performance of a two-star fund is strictly better than that of a one-star fund. Formally,*

$$\mu_2^* \mathbb{E}(e^x|F^I) + \left(1-\mu_2^*\right) \mathbb{E}(e^x|F^U) > \mu_1^* \mathbb{E}(e^x|F^I) + \left(1-\mu_1^*\right) \mathbb{E}(e^x|F^U). \tag{22}$$

The prediction stated in Corollary 4 is generally supported by prior empirical evidence. For example, Del Guercio and Tkac (2008) report that a trading strategy of investing in only five-star-rated funds results in positive risk-adjusted performance out of sample. Reuter and Zitzewitz (2015) show that in some fund classes, funds at the bottom of a higher star rating group have significantly better investment outcomes than those at the top of the next lower star rating group.[13]

C.3. Star Rating Persistence

Star ratings' predictive power is also reflected in rating persistence. We analyze rating persistence by first deriving the transition probabilities over the fund's star ratings. In an SRE, the *simple rating rule* specified in equation (20) is characterized by (x_1^*, x_2^*), and the fund's reputation is μ_1^* or μ_2^*. Hence, without exact knowledge of the fund's information status, one can calculate the probability that a two-star fund will be assigned a one-star rating in the next period as

$$p_{21} = \mu_2^* F^I\left(x_2^*\right) + \left(1-\mu_2^*\right) F^U\left(x_2^*\right). \tag{23}$$

Intuitively, if the fund has a two-star rating, its reputation is μ_2^*. That is, on average, a two-star fund is informed with probability μ_2^* and uninformed with probability $1-\mu_2^*$. Then, according to the *simple rating rule*, the fund will receive a one-star rating in the next period if and only if the investment outcome is lower than x_2^*. This occurs with probability $\mu_2^* F^I(x_2^*) + (1-\mu_2^*) F^U(x_2^*)$.

Similarly, the probability that a one-star fund will receive a two-star rating in the next period is

$$p_{12} = \mu_1^* \left(1 - F^I\left(x_1^*\right)\right) + \left(1-\mu_1^*\right)\left(1 - F^U\left(x_1^*\right)\right). \tag{24}$$

As a result, the evolution of the ratings constitutes a Markov chain with transition probability matrix

$$\mathbf{P} = (p_{ij})_{i,j\in\{1,2\}} = \begin{array}{c} \\ 1 \\ 2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} \quad 1 \quad & \quad\; 2 \quad \end{array} \\ \begin{pmatrix} 1-p_{12} & p_{12} \\ p_{21} & 1-p_{21} \end{pmatrix} \end{array}. \tag{25}$$

[13] We also note that some empirical studies, such as Blake and Morey (2000), find little statistical evidence that five-star funds outperform four-star funds. However, five-star funds do not perform worse than four-star funds. More importantly, funds with higher star ratings are larger, which may erode their productivity due to the negative size effect assumed in Berk and Green (2004). In the Internet Appendix, we analyze a perturbed model that incorporates the assumption of negative size effects. We show that while the equilibrium characterization is similar, informed funds' productivity does decline.

COROLLARY 5: *In an SRE, the two thresholds of the simple rating rule have the properties*

$$x_1^*, x_2^* \in \mathbb{R} \quad \textit{and} \quad x_1^* > x_2^*. \tag{26}$$

Therefore, in the rating transition matrix (25),

$$p_{21} \in (0, 1), p_{12} \in (0, 1), \ \textit{and} \ \ 1 - p_{21} > p_{12}. \tag{27}$$

Corollary 5 shows that star ratings are imperfectly persistent. On the one hand, the fact that $p_{21} \in (0, 1)$ implies that $1 - p_{21} \in (0, 1)$, and so a two-star fund may be downgraded to a one-star fund. On the other hand, the fact that $1 - p_{21} > p_{12}$ implies that the fraction of two-star funds that maintain a two-star rating is greater than the fraction of one-star funds that receive a two-star rating. The imperfect persistence of star ratings may reconcile the debate between the *Wall Street Journal* (*WSJ*) and Morningstar: the *WSJ* argues that the Morningstar rating is not useful to investors since a large fraction of high-rated funds have been downgraded in recent decades, while Moringstar responds that high-rated funds are still more likely to receive high ratings than lower rated funds.

IV. Star Rating Property in Other Equilibria

The equilibrium characterized in Proposition 2 features the star rating property: the fund's equilibrium reputation, as a function of the investor prior belief that is continuously distributed, can take only two values. In this section, we characterize other equilibria and explore the extent to which the star rating property holds in other equilibria.

It follows from Lemma 2 that the fund never strictly prefers to acquire information. Hence, we can partition the equilibria into two classes. In the first class of equilibria, for almost every investor prior belief $\rho \in [0, 1 - \lambda]$, the fund's information premium achieves its upper bound c, that is, the uninformed fund is indifferent between acquiring information and remaining uninformed. Therefore, in the first class of equilibria, there is almost no "shirking fund." In the second class of equilibria, the fund strictly prefers to shirk when the investor prior belief belongs to a nontrivial subset of $[0, 1 - \lambda]$. We demonstrate that the star rating property holds in both cases. Moreover, the first class of equilibria exhibits a *strong* star rating property.

A. Equilibria without Shirking Funds

In this subsection, we focus on the case in which the fund is indifferent between acquiring information and remaining uninformed at any reputation. Formally, if we denote by $\mathcal{U}$ the fund's equilibrium reputation set, the equilibrium restriction that the fund is always indifferent between acquiring information and remaining uninformed implies that[14]

$$V_I(\mu) - V_U(\mu) = c, \forall \mu \in \mathcal{U}. \tag{28}$$

The main result in this subsection is that the strong star rating property holds in all equilibria with the restriction of equation (28). Thus, a rating company that wants to accurately predict the fund's performance can group funds with the same reputation together and rate each group of funds with a certain number of stars. Therefore, the stars have a one-to-one relation with the fund's equilibrium reputation. This idea is formally presented in Proposition 3 below.

PROPOSITION 3: *If $V_I(\mu) - V_U(\mu) = c$ for all $\mu \in \mathcal{U}$ in equilibrium, then $\mathcal{U}$ includes at most countably many values.*

Proposition 3 implies that in an equilibrium in which the fund is always indifferent, its reputation is a step function of the investor prior belief. In addition, the uninformed fund will acquire information with positive probability almost surely. That is, while the uninformed fund may shirk ($\sigma(\rho) = 0$) for some investor prior beliefs, the set of such prior beliefs must have a zero Lebesgue measure on $[0, 1 - \lambda]$; otherwise, $\mathcal{U}$ would be uncountable.

The proof of Proposition 3 involves many technical arguments and thus is relegated to the Appendix. In the rest of this subsection, we provide a relatively intuitive argument. We first notice that because the fund's equilibrium reputation is weakly increasing in the investor prior belief, the function $\mu(\rho)$ has countably many discontinuities. Therefore, the range of such a function (which is the set of the fund's equilibrium reputations) can be represented as a union of at most countably many separated sets. There are two cases. First, at least one of these separated sets contains an open interval. Second, each separated set contains an isolated point only. One example of $\mathcal{U}$ is presented in Figure 6, where $\mathcal{U} = [\mu_1, \mu_2] \cup \{\mu_3\} \cup (\mu_4, \mu_5] \cup \{\mu_6\}$. Therefore, to prove Proposition 3, we just need to rule out the existence of equilibria in which there exists an open interval $\mathcal{I} \subset \mathcal{U}$ (e.g., $\mathcal{I} = (\mu_1, \mu_2)$ or $\mathcal{I} = (\mu_4, \mu_5)$ in Figure 6).

We show this by contradiction. We first decompose the information premium at any reputation $\mu \in \mathcal{U}$ into

$$\begin{aligned} V_I(\mu) - V_U(\mu) = \delta \left[\int_{-\infty}^{\infty} V_U(\mu') \mathrm{d}F^I(x) - \int_{-\infty}^{\infty} V_U(\mu') \mathrm{d}F^U(x) \right] \\ + \delta(1 - \lambda) \int_{-\infty}^{\infty} \left[V_I(\mu') - V_U(\mu') \right] \mathrm{d}F^I(x). \end{aligned} \tag{29}$$

[14] A more rigorous assumption is that the fund is indifferent between acquiring information and remaining uninformed for almost all investor prior beliefs. Our analysis and conclusions do not change under this assumption. However, for ease of exposition, we avoid the technical term "for almost all" and instead use "for all."

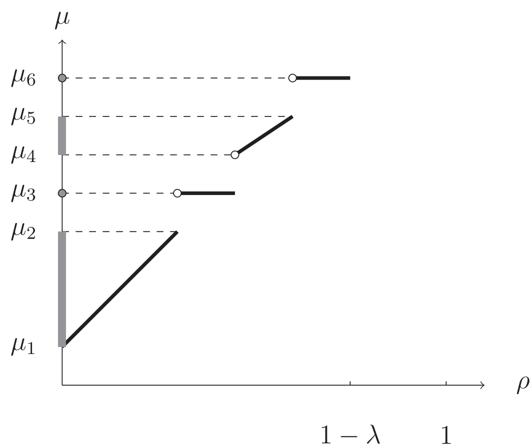


Figure 6. An example of the fund's candidate equilibrium reputation set.

The first term in equation (29) represents the benefit of being informed in the current period: it makes the fund more likely to achieve a good investment outcome and thus have a higher reputation in the next period. The second term captures the discounted future information premium for the fund. Combining (28) and (29) yields

$$\Delta_0(\mu) \triangleq \int_{-\infty}^{\infty} V_U(\mu')f^I(x)\mathrm{d}x - \int_{-\infty}^{\infty} V_U(\mu')f^U(x)\mathrm{d}x = \left[\frac{1}{\delta} - (1-\lambda)\right]c. \tag{30}$$

That is, the fund's benefit from being informed in the current period does not depend on μ and by extension the history of performance.

The key step of the proof is to show that equation (30) cannot hold for a continuum of μ's. Recall that the next-period reputation μ' is a function of the current-period reputation μ and the current investment outcome x. Then integration by substitution implies that equation (30) can be rewritten as

$$\Delta_0(\mu) = \int_{-\infty}^{\infty} \left[V_U\left(\mu'(\mu, x+1)\right) - V_U\left(\mu'(\mu, x)\right)\right] f^U(x)\mathrm{d}x = \left[\frac{1}{\delta} - (1-\lambda)\right]c.$$

Now consider two reputations $\mu_1, \mu_2 \in \mathcal{U}$. Because the next-period investor prior belief is a continuous function in both the fund's current reputation and its current investment outcome, then for any investment outcome $x \in \mathbb{R}$, there is a $\psi \in \mathbb{R}$ such that $\rho'(\mu_1, x-\psi) = \rho'(\mu_2, x)$ and therefore $\mu'(\mu_1, x-\psi) = \mu'(\mu_2, x)$.[15]

[15] From equation (9), it is clear that ψ should satisfy $\frac{\mu_1 \ell(x-\psi)}{\mu_1 \ell(x-\psi)+1-\mu_1} = \frac{\mu_2 \ell(x)}{\mu_2 \ell(x)+1-\mu_2}$, which implies that $\frac{\ell(x-\psi)}{\ell(x)} = \frac{\mu_2(1-\mu_1)}{\mu_1(1-\mu_2)}$. Since $\ell(x) = e^{\frac{2x-1}{2\phi^2}}$, we obtain $\psi = \phi^2 \ln\left[\frac{\mu_1(1-\mu_2)}{\mu_2(1-\mu_1)}\right]$.

We thus have

$$\Delta_0(\mu_2) = \int_{-\infty}^{\infty} \left[V_U\big(\mu'(\mu_1, x+1)\big) - V_U\big(\mu'(\mu_1, x)\big)\right] f^U(x+\psi)\mathrm{d}x,$$

and $\Delta_0(\mu_1) = \Delta_0(\mu_2)$ leads to

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[V_U\big(\mu'(\mu_1, x+1)\big) - V_U\big(\mu'(\mu_1, x)\big)\right] f^U(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \left[V_U\big(\mu'(\mu_1, x+1)\big) - V_U\big(\mu'(\mu_1, x)\big)\right] f^U(x+\psi)\mathrm{d}x. \end{aligned} \tag{31}$$

Equation (31) implies that the expected benefit from being informed in the current period does not change when we shift the distribution of the investment outcome by ψ.

Importantly, if the set of equilibrium reputations, $\mathcal{U}$, includes at most countably many values, equation (31) needs to hold for at most countably many ψ's. This is possible, since $V_U(\cdot)$, a function defined over $\mathcal{U}$, can take at most countably many values when $\mathcal{U}$ is countable.

By contrast, if we assume that there is an open interval $\mathcal{I} \subset \mathcal{U}$, then equation (31) must hold for a continuum of ψ's. This can be true only if $V_U(\mu'(\mu_1, x+1)) - V_U(\mu'(\mu_1, x))$ is a positive constant independent of x, or if $f^U(x+\psi)$ is invariant in ψ. However, neither of these conditions is true. The former condition cannot hold, because $V_U(\mu'(\mu_1, x+1)) - V_U(\mu'(\mu_1, x))$ converges to zero as x diverges. The latter condition violates the distribution assumption. Therefore, $\mathcal{U}$ cannot contain an open interval, and hence must include at most countably many values. That is, the star rating property necessarily arises in equilibrium if the fund is always indifferent between acquiring information and remaining uninformed.

B. Equilibria with Shirking Funds

We now analyze the case in which the fund strictly prefers not to acquire information (and so is called a *shirking fund*) when the investor prior belief ρ belongs to an open interval $\widetilde{\mathcal{V}} \subset [0, 1-\lambda]$. We first show that a fund must acquire information with positive probability for a nontrivial set of investor prior beliefs, unless the information acquisition cost is too large.

Suppose that the fund never acquires information in equilibrium, that is, $\sigma(\rho) = 0\ \forall \rho \in [0, 1-\lambda]$. It then directly follows that the equilibrium reputation $\mu = \rho$. Therefore, with the fund strategy of never acquiring information, the information premium is defined as $W_I(\rho) - W_U(\rho)$ at any investor prior belief ρ, where $W_I(\rho)$ and $W_U(\rho)$ are value functions defined in equations (12) and (13). In Figure 7 below, we numerically depict the information premium $W_I(\rho) - W_U(\rho)$ as a function of ρ when the information decay rate λ is small.

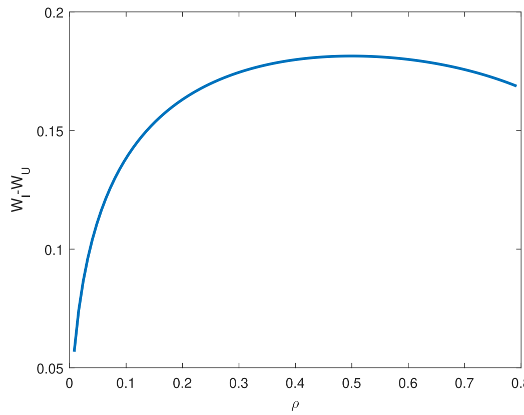


Figure 7. The information premium function $W_I(\rho) - W_U(\rho)$. (Color figure can be viewed at wileyonlinelibrary.com)

Figure 7 indicates that the information premium $W_I(\rho) - W_U(\rho)$ is small when the investor prior belief is either very low or very high. The information premium becomes larger at an intermediate prior level. Intuitively, at extreme prior beliefs, the prior dominates investor belief updating, and hence the investment outcome, as a signal of the fund's information status, has relatively weak effects on investor posterior belief. At interior prior beliefs, in contrast, the investment outcome becomes a much more informative signal, so the information premium should also be higher.

It is then easy to see that to sustain the equilibrium in which the fund never acquires information, the information acquisition cost c must be greater than or equal to $\hat{c} \equiv \max_{\rho \in [0,1-\lambda]} W_I(\rho) - W_U(\rho)$. Otherwise, there will be an open interval $\mathcal{V} \subset [0, 1-\lambda]$ in which the fund has an incentive to deviate to acquire information.

We therefore maintain the assumption that $c < \hat{c}$ and consider strategy profiles in which the fund does not acquire information when the investor prior belief belongs to a nontrivial set. Similar to Proposition 3, Proposition 4 shows that the star rating property holds under mild conditions.

PROPOSITION 4: *Suppose that there are two open subsets $\mathcal{V} \subset [0, 1-\lambda]$ and $\widetilde{\mathcal{V}} \subset [0, 1-\lambda]$, such that*

$$W_I(\rho) - W_U(\rho) \begin{cases} = c, & \forall \rho \in \mathcal{V} \\ < c, & \forall \rho \in \widetilde{\mathcal{V}}. \end{cases} \tag{32}$$

Suppose further that $\widetilde{\mathcal{V}}$ contains an open neighborhood of $\rho = 0$, an open neighborhood of $\rho = 1 - \lambda$, or both.[16] *Then the set $\mu(\mathcal{V})$ includes at most countably many values.*

[16] This assumption excludes the case in which the fund acquires information with positive probability when the investor prior belief is close to both 0 and $1 - \lambda$ but does not acquire information

In the rest of this section, we provide further characterization of equilibria with shirking funds. Relative to the SRE characterized in Section III, characterization of an equilibrium with shirking funds is much more complicated. On the one hand, because the fund does not acquire information when the investor prior belief belongs to some open intervals, analytically characterizing its value function at each reputation is intractable. On the other hand, an equilibrium may feature an alternative "shirking" and "working" strategy. Specifically, there may be a partition $\{\mathcal{V}_n\}$ of the prior belief space $[0, 1-\lambda]$ such that the fund does not acquire information for all $\rho \in \mathcal{V}_n$ when n is odd, but acquires information with positive probability for all $\rho \in \mathcal{V}_n$ when n is even. We therefore provide numerical equilibrium analyses. Since the information acquisition cost will affect the equilibrium characterization, we study both the case of a sufficiently small information acquisition cost and the case of an intermediate information acquisition cost.

B.1. Small Information Acquisition Cost

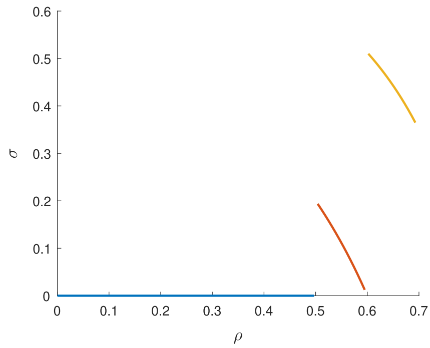
We first study the information premium at two extreme investor prior beliefs, $\rho = 0$ and $\rho = 1-\lambda$. At $\rho = 0$, if investors believe that the fund does not acquire information, then the fund will have the reputation $\mu = 0$. In this case, the next-period investor prior belief will be independent of the fund's current investment outcome, which implies that the next-period investor prior belief will still be zero. Therefore, if investors believe that the fund does not acquire information at $\rho = 0$, the information premium will be $W_I(0) - W_U(0) = 0$. Moreover, because the distribution of informed investment and the distribution of uninformed investment have identical support, and the fund's information status is unobservable, both W_I and W_U are continuous at $\rho = 0$. Therefore, there may exist an equilibrium in which the fund does not acquire information when the investor prior belief is very low.

At the other extreme, at $\rho = 1-\lambda$, the information premium will be bounded away from zero, if investors believe that the fund does not acquire information. In such a case, the fund's reputation is $\mu = 1-\lambda$, and so a better investment outcome can lead to a higher next-period reputation and thus larger future cash flows. Therefore, when the information acquisition cost c is sufficiently small, the fund with $\rho = 1-\lambda$ will acquire information with positive probability. Because W_I and W_U are also continuous at $\rho = 1-\lambda$, we get Lemma 3 below.

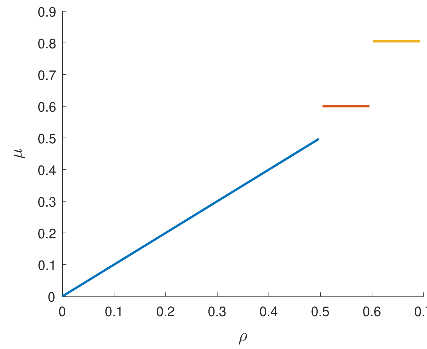
LEMMA 3: *When the information acquisition cost c is sufficiently small, there exists $\bar{\rho} \in (0, 1-\lambda)$ such that, in equilibrium, the funds must be acquiring information with positive probability when $\rho \in (\bar{\rho}, 1-\lambda]$.*

Lemma 3 shows that when the cost of information acquisition is sufficiently small, shirking is not sustainable when the investor prior belief is close to the

at some interior level of investor prior beliefs. The intuition also follows from the fact that at interior investor prior beliefs, the investment outcome is a more informative signal, and hence, the information premium is larger and the fund has stronger incentives to acquire information. This is confirmed by our unsuccessful attempts to numerically construct an equilibrium with this property.



Panel A. Information Acquisition Strategy



Panel B. Reputation

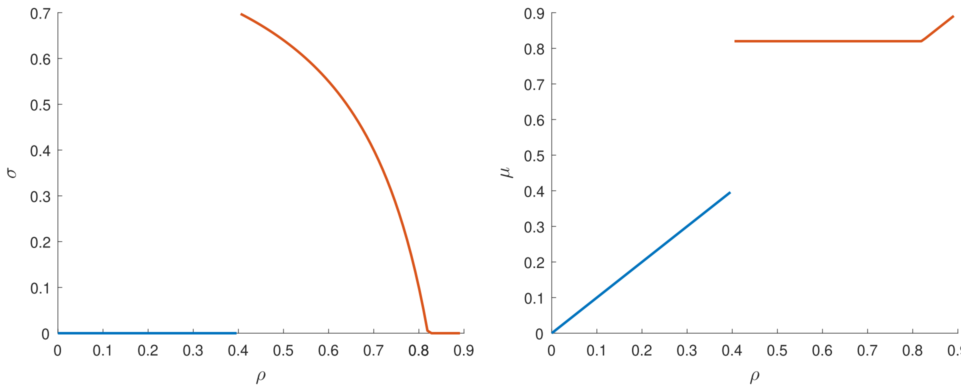
Figure 8. An equilibrium with three groups. (Color figure can be viewed at wileyonlinelibrary.com)

upper bound $1-\lambda$. This is intuitive. The information premium $W_I(\rho) - W_U(\rho)$ is bounded below from zero at $\rho = 1-\lambda$ if investors believe that the fund does not acquire information. Therefore, as long as the information acquisition cost $c < W_I(1-\lambda) - W_U(1-\lambda)$, shirking ($\sigma(\rho) = 0$) is not incentive-compatible when $\rho \in (\bar{\rho}, 1-\lambda)$ for some $\bar{\rho} < 1-\lambda$ due to the continuity of value functions.

Given Proposition 4 and Lemma 3, we numerically construct an equilibrium with three groups. Here, we replace "star" by "group" because in some groups, the funds may not have the same reputation. Specifically, the funds in the first group have investor prior beliefs below $\bar{\rho}$ and hence do not acquire information. Proposition 4 then suggests that funds in the second group have an identical reputation $\bar{\mu}_2 > \bar{\rho}$, and funds in the third group have an identical reputation $\bar{\mu}_3 > \bar{\mu}_2$.

Figure 8 illustrates such an equilibrium. Specifically, Panels A and B of Figure 8 show the fund's information acquisition strategy and equilibrium reputation, respectively. When the investor prior belief is below the threshold $\bar{\rho}$, the fund does not acquire information and hence its reputation is the same as its investor prior belief. In the second group, the fund acquires information with positive probability, but the probability of information acquisition is decreasing, such that all funds in this group have the same reputation. The funds in the third group behave like those in the second group, but their reputation is strictly higher.

We can compare an equilibrium with three groups to an SRE with two stars by comparing Figure 8 to Figures 2 and 3. First, the second and third groups in the equilibrium with three groups have properties similar to those in the SRE with two stars. Specifically, funds within one group have the same reputation, and fund size and average productivity experience discrete jumps at the threshold in the investor prior belief space that separates the second and third groups. Second, an equilibrium with three groups differs from an SRE with two stars mainly in that funds in the first group do not acquire information. Therefore, there is an open interval as a component of the equilibrium reputation set.



Panel A. Information Acquisition Strategy

Panel B. Reputation

Figure 9. An equilibrium with three groups. (Color figure can be viewed at wileyonlinelibrary.com)

The equilibrium with three groups may help explain another empirical observation: the lowest-rated funds persistently perform poorly. For example, Blake and Morey (2000) find that low ratings from Morningstar generally indicate relatively poor future performance. Also, a Morningstar research report published on October 25, 2017, shows that funds assigned a one-star rating have a significantly higher liquidation fraction in the next 10 years. In the equilibrium with three groups in our model, the funds in the lowest ranked group do not acquire information and hence when their information becomes obsolete, their investment outcomes become worse and worse, on average, and are ultimately liquidated because of very low cash flows.

B.2. Intermediate Information Acquisition Cost

When the information acquisition cost c is at an intermediate level, the information premium at $\rho = 1 - \lambda$ may be strictly less than the information acquisition cost. Hence, the fund may choose not to acquire information when the investor prior belief is either very low or very high. However, Proposition 4 implies that when the fund acquires information with positive probability, it has at most countably many reputations.

We next numerically construct another equilibrium with three groups and depict it in Figure 9. We classify the funds with $\rho \leq \hat{\rho}_1$ into the first group, the funds with $\rho \in (\hat{\rho}_1, \hat{\rho}_2]$ into the second group, and the funds with $\rho > \hat{\rho}_2$ into the third group. Also, the funds in the first and third groups do not acquire information, and so their reputations are the same as the investor prior beliefs. Only the funds in the second group acquire information, and the probability that they do so is decreasing in ρ, such that all second-group funds have the same reputation.

V. Conclusions

This paper provides a rational theory of a mutual fund's dynamic incentives to acquire information for the documented star rating effects often attributed to investors' irrationality, such as limited attention. We show that the star rating property arises in equilibrium in our rational model, and thus investors may rationally respond to star ratings. We then derive a number of equilibrium properties and discuss how they are consistent with empirical observations.

Our theory also provides a rational explanation for the coarseness of the prevailing rating systems. Because of the star rating property, a rating company that wants to accurately predict a fund's performance can group funds with the same reputation together and rate each group with a certain number of stars. Since these stars have a one-to-one relation with the fund's reputation, they play a role in indicating the fund's reputation. In equilibrium, investors make investment decisions based on the fund's reputation, and therefore, the coarseness of a star rating system does not lead to any information loss, even though the fund's past performance is continuously distributed.

Finally, we provide a new way of looking at a mutual fund's reputation. We define a fund's reputation as investors' belief about whether the fund is informed. Because information can be endogenously acquired but can decay stochastically, the fund needs to exert effort to build and maintain its reputation. Therefore, the fund's reputation becomes one of its important assets. Our view of reputation challenges the traditional one that holds that the fund's reputation depends on the manager's innate and unchangeable ability and hence cannot be enhanced further.

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Appendix A: Proofs

In this appendix, we present proofs.

PROOF OF LEMMA 1: Note that the fund's investment outcome x_t is normally distributed, regardless of whether the fund is informed. Hence, by a normal random variable's moment-generating function, we have

$$\mathbb{E}(e^{x_t}|F^I) = e^{1+\frac{1}{2}\phi^2},$$
$$\mathbb{E}(e^{x_t}|F^U) = e^{\frac{1}{2}\phi^2}.$$

Thus, with reputation μ_t, any investor's expected after-fee return from buying the fund's shares is

$$\mu_t\mathbb{E}(e^{x_t}|F^I) + (1-\mu_t)\mathbb{E}(e^{x_t}|F^U) = \mu_t e^{1+\frac{1}{2}\phi^2} + (1-\mu_t)e^{\frac{1}{2}\phi^2}.$$

Similarly, if investor i decides to choose his outside option in period t, the expected gross rate of return is

$$\mathbb{E}(e^{\tilde{x}_{it}}|\bar{x}_i) = e^{\bar{x}_i + \frac{1}{2}\phi^2}.$$

In each period t, investor i wants to maximize his end-of-period wealth and thus will buy the fund's shares if and only if

$$\mu_t e + (1 - \mu_t) \geq e^{\bar{x}_i},$$

which is equivalent to

$$\ln\left[\mu_t e + (1 - \mu_t)\right] \geq \bar{x}_i.$$

It follows from the assumption $\bar{x}_i \sim U[0, 1]$ that the fund's cash flow in period t is

$$\theta_t = \ln\left[\mu_t e + (1 - \mu_t)\right].$$

□

PROOF OF LEMMA 2: Suppose that $\sigma(\rho) = 1$ for some investor prior belief ρ. The fund with the investor prior belief ρ will have reputation $\mu = \rho + (1 - \rho)\sigma(\rho) = 1$. As a consequence, the investor prior belief in the next period will be $1 - \lambda$, independent of the investment outcome.

To support this equilibrium, it must be the case that $W_I(\rho) - W_U(\rho) \geq c$, which, from equations (12) and (13), implies that

$$W_I(1 - \lambda) - \max\left\{W_I(1 - \lambda) - c, W_U(1 - \lambda)\right\} \geq \frac{c}{\delta(1 - \lambda)}. \tag{A.1}$$

Here, $W_K(1 - \lambda)$ is the value of the function $W_K(\cdot)$ evaluated at $1 - \lambda$ ($K = I, U$).

If $W_I(1 - \lambda) - c > W_U(1 - \lambda)$, equation (A.1) becomes $W_I(1 - \lambda) - [W_I(1 - \lambda) - c] = c \geq \frac{c}{\delta(1-\lambda)}$. Since $\delta < 1$ and $1 - \lambda < 1$, this equation obviously does not hold. In the case in which $W_I(1 - \lambda) - c \leq W_U(1 - \lambda)$, equation (A.1) becomes $W_I(1 - \lambda) - W_U(1 - \lambda) \geq \frac{c}{\delta(1-\lambda)} > c$. As a result, $\sigma(1 - \lambda) = 1$ in the equilibrium under consideration. It follows that $W_I(1 - \lambda) - W_U(1 - \lambda) = \delta(1 - \lambda)[W_I(1 - \lambda) - W_U(1 - \lambda)]$, which is a contradiction.

We therefore conclude that equation (A.1) can never hold, and hence $\sigma(\rho) < 1$ for all ρ. □

PROOF OF PROPOSITION 1: The proof follows the standard argument in Stokey and Lucas (1989). Denote by $\mathcal{B}$ the set of bounded functions $f : [0, 1 - \lambda] \to \mathbb{R}$. By Example 17.14 in Sutherland (2009), the metric space $(\mathcal{B}, ||\cdot||)$ is complete, where $||\cdot||$ represents the standard sup norm. Define a mapping Γ on $\mathcal{B}$ as

$$\Gamma(\hat{V}_U) = \pi\hat{\theta}(\mu) + \delta \int \hat{V}_U(\mu'(\mu, x))\mathrm{d}F^U(x),$$

where $\mu'(\mu, x)$ is specified in the main text. It follows directly from the standard Blackwell sufficient conditions that Γ is a contraction mapping, and thus $\Gamma(\cdot)$

has a unique fixed point. Let $\mathcal{B}'$ denote the set of bounded, nondecreasing functions on $[0, 1-\lambda]$ and $\mathcal{B}''$ denote the set of bounded, strictly increasing functions on $[0, 1-\lambda]$. Obviously, $\mathcal{B}'' \subset \mathcal{B}' \subset \mathcal{B}$ and $\mathcal{B}'$ is a closed subset of $\mathcal{B}$. Since we focus on a monotone equilibrium, $\mu'(\mu, x)$ is nondecreasing in μ for each x. In addition, θ is strictly increasing in μ. Simple algebra shows that $\Gamma(\mathcal{B}') \subseteq \mathcal{B}''$. By Corollary 1 in chapter 4 of Stokey and Lucas (1989), we conclude that $V_U(\mu)$, the unique fixed point of $\Gamma(\cdot)$, is strictly increasing in μ. Similarly, one can prove that $V_I(\mu)$ is strictly increasing in μ.

For the second part of this proposition, $V_I(\mu) - V_U(\mu) \geq 0$ holds trivially, because at any reputation level, the informed fund's investment outcomes first-order stochastically dominate those of the uninformed fund. We then suppose that there is a reputation level $\mu' \in (0, 1)$ such that $V_I(\mu') - V_U(\mu') > c$. There must be an investor prior belief ρ such that $\mu(\rho) = \mu'$ and $W_I(\rho) - W_U(\rho) > c$, implying that at ρ, the fund strictly prefers to acquire information. But this violates the conclusion in Lemma 2. Therefore, $0 \leq V_I(\mu) - V_U(\mu) \leq c, \forall \mu \in (0, 1)$. □

PROOF OF PROPOSITION 2: First, since reputation is increasing in the investor prior belief, we must have $\mu_2^* \geq \mu_1^*$. Notice that if $\mu_2^* = \mu_1^*$, the fund would strictly prefer not to acquire information, because it will always get a constant cash flow in each period regardless of its investment outcome. So, it must be the case that $\mu_2^* > \mu_1^*$.

From our requirement of monotone equilibrium, there exists $\hat{\rho} \leq \mu_1^*$ such that

$$\mu(\rho) = \begin{cases} \mu_1^* & \text{if } \rho \leq \hat{\rho} \\ \mu_2^* & \text{if } \rho > \hat{\rho}. \end{cases}$$

Define x_j^* as the investment outcome such that if the fund with reputation μ_j^* gets the investment outcome x_j^*, then by equation (A.16), the next-period investor prior belief is

$$\hat{\rho} = \frac{(1-\lambda)\ell\left(x_j^*\right)\mu_j^*}{\ell\left(x_j^*\right)\mu_j^* + 1 - \mu_j^*}$$

for $j = 1, 2$. It is obvious that $x_1^* > x_2^*$ as $\mu_2^* > \mu_1^*$. Because $\ell(\cdot)$ is strictly increasing, $\mu(\mu_j^*, x) = \mu_2^*$ if and only if $x \geq x_j^*$.

Note that in equations (12) and (13), the investor prior belief determines the fund's value functions through the fund's reputations only. Hence, given the restrictions that the fund has only two possible reputations, μ_1^* and μ_2^*, and that the fund is mixing and thus must be indifferent between being informed or remaining uninformed, we can rewrite the fund's value functions as

$$\begin{aligned} V_I^1 = \pi\theta_1^* + \delta\Big[&\Big(1 - F^I\big(x_1^*\big)\Big)(1-\lambda)V_I^2 \\ &+ \Big(1 - F^I\big(x_1^*\big)\Big)\lambda V_U^2 + F^I\big(x_1^*\big)(1-\lambda)V_I^1 + F^I\big(x_1^*\big)\lambda V_U^1\Big], \end{aligned} \tag{A.2}$$

$$V_U^1 = \pi\theta_1^* + \delta\Big[\Big(1 - F^U\big(x_1^*\big)\Big)V_U^2 + F^U\big(x_1^*\big)V_U^1\Big], \tag{A.3}$$

$$\begin{aligned} V_I^2 = \pi\theta_2^* + \delta\Big[&\Big(1 - F^I\big(x_2^*\big)\Big)(1-\lambda)V_I^2 \\ &+ \Big(1 - F^I\big(x_2^*\big)\Big)\lambda V_U^2 + F^I\big(x_2^*\big)(1-\lambda)V_I^1 + F^I\big(x_2^*\big)\lambda V_U^1\Big], \end{aligned} \tag{A.4}$$

$$V_U^2 = \pi\theta_2^* + \delta\Big[\Big(1 - F^U\big(x_2^*\big)\Big)V_U^2 + F^U\big(x_2^*\big)V_U^1\Big], \tag{A.5}$$

where $\theta_j^* = \hat{\theta}(\mu_j^*(\rho))$ for $j = 1, 2$. Notice that equations (A.2) to (A.5) are determined by three unknowns $\{\mu_1^*, \mu_2^*, \hat{\rho}\}$.

The rest of the proof proceeds in two steps. In the first step, we derive the explicit expression of $\bar{c} > 0$, and in the second step, we construct an equilibrium by specifying a triple $(\{\mu_1^*, \mu_2^*, \hat{\rho}\})$ for each $c < \bar{c}$.

Step 1: *Expression of* $\bar{c}$. In this step, we first express x_1^* and x_2^* as functions of μ_1^* and μ_2^*, which enables us to construct an equilibrium by solving μ_1^* and μ_2^*. Because the uninformed fund is playing a mixed strategy, it must be indifferent between acquiring information and remaining uninformed for all investor prior beliefs. As a result,

$$c = V_I^1 - V_U^1 = V_I^2 - V_U^2,$$

which, together with equations (A.2) to (A.5), implies that

$$\begin{aligned} \left[\frac{1}{\delta} - (1-\lambda)\right]c &= \Big[F^U\big(x_1^*\big) - F^I\big(x_1^*\big)\Big]\big(V_U^2 - V_U^1\big) \\ &= \Big[F^U\big(x_2^*\big) - F^I\big(x_2^*\big)\Big]\big(V_U^2 - V_U^1\big). \end{aligned} \tag{A.6}$$

Equation (A.6) implies that

$$F^U\big(x_1^*\big) - F^I\big(x_1^*\big) = F^U\big(x_2^*\big) - F^I\big(x_2^*\big). \tag{A.7}$$

Since F^U and F^I are normal distribution functions with means 0 and 1, respectively, equation (A.7) implies that $x_2^* = 1 - x_1^*$.[17] Moreover, the likelihood function satisfies

$$\ell(x) = \frac{f^I(x)}{f^U(x)} = e^{\frac{2x-1}{2\phi^2}}. \tag{A.8}$$

[17] Another solution to equation (A.7) is $x_1^* = x_2^*$. However, in a monotone equilibrium, we have $\mu_2^* > \mu_1^*$, and so $x_1^* > x_2^*$, and thus, such a solution cannot be part of an equilibrium.

Therefore, $x_2^* = 1 - x_1^*$ implies that $\ell_2 = 1/\ell_1$, where $\ell_1 = \ell(x_1^*)$ and $\ell_2 = \ell(x_2^*)$. Equations (5) and (6) imply that

$$\hat{\rho} = (1-\lambda)\frac{\ell_2\mu_2^*}{\ell_2\mu_2^* + (1-\mu_2^*)} = (1-\lambda)\frac{\ell_1\mu_1^*}{\ell_1\mu_1^* + (1-\mu_1^*)}. \tag{A.9}$$

Substituting $\ell_2 = 1/\ell_1$ into equation (A.9), we get

$$\hat{\rho} = (1-\lambda)\frac{\mu_2^*}{\mu_2^* + \ell_1\big(1-\mu_2^*\big)}.$$

This equation, together with equation (A.9), yields

$$\ell_1 = \sqrt{\frac{(1-\mu_1^*)\mu_2^*}{\mu_1^*\big(1-\mu_2^*\big)}}, \tag{A.10}$$

which implies that x_1^*, x_2^*, and $\hat{\rho}$ can be expressed as a function of μ_1^* and μ_2^*.
Substituting the requirement $\mu_1^* \geq \hat{\rho}$ into equation (A.9) for one-star funds, we obtain

$$(1-\lambda)\ell_1 \leq \ell_1\mu_1^* + 1 - \mu_1^*. \tag{A.11}$$

Substituting V_U^1 in equation (A.3) and V_U^2 in equation (A.5) into equation (A.6), we have

$$c = \Omega(\mu_1^*, \mu_2^*) \triangleq \frac{\delta}{1-\delta(1-\lambda)} \frac{\pi(\theta_2^* - \theta_1^*)\big[F^U\big(x_1^*\big) - F^I\big(x_1^*\big)\big]}{1-\delta\big[F^U\big(x_1^*\big) - F^U\big(x_2^*\big)\big]}. \tag{A.12}$$

To put an upper bound on c, we want to derive the maximum of $\Omega(\mu_1^*, \mu_2^*)$ subject to the feasibility constraints of (μ_1^*, μ_2^*). Notice that by equation (A.10), $\mu_1^* < \mu_2^*$ implies that $\ell_1 > 1$ and hence equation (A.11) can be rewritten as

$$\mu_1^* \geq 1 - \frac{\lambda\ell_1}{\ell_1 - 1}.$$

Next, define $\bar{c}$ such that

$$\begin{aligned} &\bar{c} = \max_{\mu_1,\mu_2} \Omega(\mu_1, \mu_2) \\ &\text{s.t. } (1-\lambda) \leq \mu_2 \leq 1, \\ &\max\left\{0, 1 - \frac{\lambda\ell_1(\mu_1,\mu_2)}{\ell_1(\mu_1,\mu_2) - 1}\right\} \leq \mu_1 \leq \mu_2. \end{aligned} \tag{A.13}$$

Here, $\bar{c}$ is well defined, because $\Omega(\mu_1, \mu_2)$ is a continuous function in (μ_1, μ_2), and the constraints constitute a compact set of (μ_1, μ_2).

Step 2: *Construction of an equilibrium for $c < \bar{c}$.* First, notice that the constraints of equation (A.13),

$$(1-\lambda) \le \mu_2 \le 1,\ \max\left\{0, 1 - \frac{\lambda \ell_1(\mu_1, \mu_2)}{\ell_1(\mu_1, \mu_2) - 1}\right\} \le \mu_1 \le \mu_2,$$

define a connected set on the two-dimensional space of (μ_1, μ_2). Obviously, μ_2 lies in a compact and connected set $[1-\lambda, 1]$. For μ_1, we can prove that there exists $0 \le \hat{\mu}_1(\mu_2) < \mu_2$ such that the constraint $\max\{0, 1 - \frac{\lambda \ell_1(\mu_1,\mu_2)}{\ell_1(\mu_1,\mu_2)-1}\} \le \mu_1 \le \mu_2$ is satisfied if and only if $\hat{\mu}_1(\mu_2) \le \mu_1 \le \mu_2$. This is because $\ell_1(\mu_1, \mu_2)$ is decreasing in μ_1 from equation (A.10), which implies that $1 - \frac{\lambda \ell_1(\mu_1,\mu_2)}{\ell_1(\mu_1,\mu_2)-1}$ is decreasing in μ_1. Hence, if $1 - \frac{\lambda \ell_1(\mu_1,\mu_2)}{\ell_1(\mu_1,\mu_2)-1} = \mu_1$ at some $\hat{\mu}_1$, we must have $1 - \frac{\lambda \ell_1(\mu_1,\mu_2)}{\ell_1(\mu_1,\mu_2)-1} < \mu_1$ for $\mu_1 > \hat{\mu}_1$. Moreover, $\hat{\mu}_1(\mu_2) < \mu_2$ because when $\mu_1 = \mu_2$, by equation (A.10), $\ell_1 = 1$ and the constraint $\max\{0, 1 - \frac{\lambda \ell_1(\mu_1,\mu_2)}{\ell_1(\mu_1,\mu_2)-1}\} \le \mu_1$ is naturally satisfied.

Since $\Omega(\mu_1, \mu_2)$ is a continuous function in (μ_1, μ_2) that is defined on a connected set, the range of $\Omega(\mu_1, \mu_2)$ must be connected (i.e., an interval). Since the minimum of $\Omega(\mu_1, \mu_2)$ is zero when $\mu_1 = \mu_2$, we conclude that the range of Ω is $[0, \bar{c}]$. Therefore, for any $c \in (0, \bar{c})$, there exists a pair $\{\mu_1^*, \mu_2^*\}$ that satisfies $(1-\lambda) \le \mu_2^* < 1$ and $\max\{0, 1 - \frac{\lambda \ell_1(\mu_1^*,\mu_2^*)}{\ell_1(\mu_1^*,\mu_2^*)-1}\} \le \mu_1^* < \mu_2^*$, and that solves equation (A.12).[18]

Given $\{\mu_1^*, \mu_2^*\}$, we define $\{x_1^*, x_2^*, \hat{\rho}\}$ as follows:

$$\ell(x_1^*) = \ell_1 = \sqrt{\frac{(1-\mu_1^*)\mu_2^*}{\mu_1^*(1-\mu_2^*)}},$$

$$x_2^* = 1 - x_1^* \quad \text{and} \quad \hat{\rho} = (1-\lambda)\frac{\ell_1 \mu_1^*}{\ell_1 \mu_1^* + (1-\mu_1^*)}.$$

We want to show that the triple $\{\hat{\rho}, x_1^*, x_2^*\}$ constructed above and the fund's reputations $\{\mu_1^*, \mu_2^*\}$ constitute an equilibrium. It follows from equations (A.3) and (A.5) that

$$V_U^2 - V_U^1 = \frac{\pi(\theta_2^* - \theta_1^*)}{1 - \delta\left[F^U(x_1^*) - F^U(x_2^*)\right]}.$$

Equation (A.6) is naturally satisfied since $\Omega(\mu_1^*, \mu_2^*) = c$. This implies that

$$c = V_I^1 - V_U^1 = V_I^2 - V_U^2.$$

[18] We change the constraint $\mu_2^* \le 1$ in equation (A.13) to $\mu_2^* < 1$ because $\Omega(\mu_1^*, \mu_2^*) = 0$ when $\mu_2^* = 1$. The reason is as follows. If $\mu_1^* < \mu_2^* = 1$, then $\ell_1 = +\infty$ by equation (A.10). Therefore, it must be the case that $x_1^* = +\infty$, and hence, $\Omega(\mu_1^*, \mu_2^*) = 0$ since $F^U(x_1^*) - F^I(x_1^*) = 0$.

Therefore, an uninformed fund will be indifferent between acquiring information and remaining uninformed, implying that the fund has no incentive to deviate from the equilibrium strategy (18). It follows that the proposed strategy profile is indeed an equilibrium. □

PROOF OF COROLLARY 1: Denote

$$\mathcal{M} = \left\{(\mu_1, \mu_2) : (1-\lambda) \le \mu_2 \le 1,\ \max\left\{0, 1 - \frac{\lambda \ell_1(\mu_1, \mu_2)}{\ell_1(\mu_1, \mu_2) - 1}\right\} \le \mu_1 \le \mu_2\right\},$$

and suppose that $\Omega(\mu_1, \mu_2)$ is maximized at $(\bar{\mu}_1, \bar{\mu}_2) \in \mathcal{M}$, that is, $\Omega(\bar{\mu}_1, \bar{\mu}_2) = \bar{c}$. $\mathcal{M}$ has a subset

$$\hat{\mathcal{M}} = \{(\mu_1, \mu_2) \in \mathcal{M} : \mu_2 = 1 \text{ or } \mu_1 = \mu_2\}.$$

From our proof of Proposition 2, $\Omega(\mu_1, \mu_2) = 0$ for all $(\mu_1, \mu_2) \in \hat{\mathcal{M}}$. For each $(\mu_1, \mu_2) \in \hat{\mathcal{M}}$, we can find a line $L_{(\mu_1,\mu_2)}$ connecting $(\bar{\mu}_1, \bar{\mu}_2)$ and (μ_1, μ_2). Obviously, the entire line $L_{(\mu_1,\mu_2)}$ is contained in $\mathcal{M}$, as $\mathcal{M}$ is a convex set.[19] There is a continuum of $L_{(\mu_1,\mu_2)}$ as there is a continuum of $(\mu_1, \mu_2) \in \hat{\mathcal{M}}$. And on each $L_{(\mu_1,\mu_2)}$, Ω changes continuously from $\bar{c}$ to 0. Therefore, for each $c < \bar{c}$, we can find some $(\mu_1^*, \mu_2^*) \in \hat{\ell}_{(\mu_1,\mu_2)}$ such that $\Omega(\mu_1^*, \mu_2^*) = c$. Since there is a continuum of $L_{(\mu_1,\mu_2)}$, there is a continuum of (μ_1^*, μ_2^*) satisfying $\Omega(\mu_1^*, \mu_2^*) = c$, each of which corresponds to an SRE with two stars. □

PROOF OF COROLLARY 2: In the proof of Proposition 2, we show that $\mu_2^* > \mu_1^*$. It then follows from Lemma 1 that if a fund has rating $\iota_t = 1$ in period t and is assigned rating $\iota_{t+1} = 2$ in period $t+1$, it will receive cash flow

$$\ln\left[\mu_2^* e + \left(1 - \mu_2^*\right)\right] - \ln\left[\mu_1^* e + \left(1 - \mu_1^*\right)\right],$$

which is strictly positive. □

PROOF OF COROLLARY 4: Because F^I strictly first-order stochastically dominates F^U and $\mu_2^* > \mu_1^*$, we have

$$\left(\mu_2^* - \mu_1^*\right)\left[\mathbb{E}(x|F^I) - \mathbb{E}(x|F^U)\right] > 0,$$

which proves the result. □

PROOF OF COROLLARY 5: From the proof of Proposition 2, we have $\mu_2^* > \mu_1^*$. Then by equation (A.10), we have $\ell_1 > 1$, which by equation (A.8) implies $x_1^* > \frac{1}{2}$, and hence $x_2^* = 1 - x_1^* < \frac{1}{2} < x_1^*$. Therefore, $F^K(x_1^*) > F^K(x_2^*)$ for $K = I, U$.

[19] The convexity of $\mathcal{M}$ comes from the fact that, by definition, $\mathcal{M}$ can be further written as

$$\mathcal{M} = \{(\mu_1, \mu_2) : \mu_1 \le 1 - \lambda,\ (1-\lambda) \le \mu_2 \le \hat{\mu}_2(\mu_1)\} \bigcup \{(\mu_1, \mu_2) : \mu_1 > 1 - \lambda,\ \mu_1 \le \mu_2 \le 1\},$$

where $\hat{\mu}_2(\mu_1)$ is given by $1 - \frac{\lambda \ell_1(\mu_1, \hat{\mu}_2)}{\ell_1(\mu_1, \hat{\mu}_2) - 1} = \mu_1$. It is straightforward to verify that $\hat{\mu}_2(\mu_1)$ is a concave function, and hence, $\mathcal{M}$ is convex.

Equation (23) implies that the probability of a two-star fund maintaining its two-star rating is

$$p_{22} = 1 - p_{21} = \mu_2^*\Big(1 - F^I(x_2^*)\Big) + (1 - \mu_2^*)\Big(1 - F^U(x_2^*)\Big).$$

The conclusions that $p_{12} \in (0, 1)$ and $p_{21} \in (0, 1)$ follow from the unbounded likelihood ratio property of the normal distribution. Furthermore, because $F^I(x_j^*) < F^U(x_j^*)$, we have

$$\begin{aligned} p_{22} &= \mu_2^*\Big(1 - F^I(x_2^*)\Big) + (1 - \mu_2^*)\Big(1 - F^U(x_2^*)\Big) \\ &> \mu_2^*\Big(1 - F^I(x_1^*)\Big) + (1 - \mu_2^*)\Big(1 - F^U(x_1^*)\Big) \\ &> \mu_1^*\Big(1 - F^I(x_1^*)\Big) + (1 - \mu_1^*)\Big(1 - F^U(x_1^*)\Big) \\ &= p_{12}. \end{aligned}$$

□

PROOF OF PROPOSITION 3: Since we focus on a monotone equilibrium, the equilibrium reputation $\mu(\rho)$ specified in equation (7) is weakly increasing, so it has at most countably many discontinuities. As a result, the set $\mathcal{U} \equiv \mu([0, 1 - \lambda])$ can be represented as a union of at most countably many disjoint connected sets, $\cup_i \mathcal{U}_i$. Here, for each i, either $\mathcal{U}_i$ is a singleton, or there exists an open interval $\mathcal{I} \subset \mathcal{U}_i$. Therefore, to prove Proposition 3, it is sufficient to show that no $\mathcal{U}_i$ can contain an open interval $\mathcal{I}$. We prove this by contradiction. Suppose that such an open interval $\mathcal{I} \subset \mathcal{U}$ exists. Then equation (30) holds for each $\mu \in \mathcal{I}$.

As function $\mu(\rho)$ has at most countably many discontinuities, we denote by $\{\rho_n\}_{n=1}^N$, with $\rho_1 < \cdots < \rho_N$, discontinuous points of $\mu(\rho)$, where $N \in \{0, 1, \cdots, \infty\}$ is the total number of discontinuous points. Our proof holds for any N. Because $\ell(x)$ is continuous in x and ranges from 0 to ∞, then for any $\mu \in \mathcal{I}$, the next-period prior belief ρ' can range from 0 to $1 - \lambda$. Therefore, there exist at most countably many investment outcomes $\{x_n(\mu)\}_{n=1}^N$ with $x_1(\mu) < \cdots < x_N(\mu)$ satisfying $\rho_n = \frac{(1-\lambda)\ell(x_n)\mu}{\ell(x_n)\mu+1-\mu}$ such that the next-period reputation μ' is discontinuous in x at these investment outcomes. We can thus rewrite $\Delta_0(\mu)$ as

$$\Delta_0(\mu) \triangleq \int_{-\infty}^{x_1(\mu)} V_U(\mu')\mathrm{d}[F^I(x) - F^U(x)] + \cdots + \int_{x_N(\mu)}^{\infty} V_U(\mu')\mathrm{d}[F^I(x) - F^U(x)]. \quad \text{(A.14)}$$

The rest of the proof proceeds in four steps. In the first three steps, we establish some properties that the fund's equilibrium value function must satisfy. Then, in step 4, we argue that these properties lead to a contradiction.

Step 1. Define

$$\Delta_1(\mu) \triangleq \int_{-\infty}^{\infty} V_U(\mu'(\mu,x))\Big[(x-1)f^I(x) - xf^U(x)\Big]\mathrm{d}x.$$

We show that $\Delta_1(\mu) = 0$ for all $\mu \in \mathcal{I}$.
Take the derivative of equation (A.14) with respect to μ. Notice that μ enters equation (A.14) by affecting both $x_n(\mu)$ and $V_U(\cdot)$. On each interval $(x_n(\mu), x_{n+1}(\mu))$ for $n = 0, \dots, N$ with $x_0(\mu) = -\infty$ and $x_{N+1}(\mu) = \infty$, $\mu'(\mu,x)$ is a continuous function of x, which implies two cases as shown by Figure 6: either μ' is an isolated point like μ_3 and μ_6 in Figure 6, or μ' is contained in an interval. Let $\mathcal{C}$ be the set of nonisolated points in $\mathcal{U}$. In other words, μ' changes continuously with ρ' for $\mu' \in \mathcal{C}$. Obviously, the set $\mathcal{C}$ is nonempty since $\mathcal{I} \subset \mathcal{C}$. So, if Δ_0 is differentiable with respect to μ, we can write

$$\begin{aligned} 0 = \Delta_0'(\mu) = &\int_{\mu' \in \mathcal{C}} V_U'(\mu')\frac{\partial \mu'}{\partial \mu}\mathrm{d}[F^I(x) - F^U(x)] \\ &- \sum_{n=1}^{N}(V_U(\tilde{\mu}_n) - V_U(\hat{\mu}_n))\Big[f^I(x_n(\mu)) - f^U(x_n(\mu))\Big]\frac{\partial x_n(\mu)}{\partial \mu}, \end{aligned} \tag{A.15}$$

where the reputation μ jumps from $\hat{\mu}_n$ to $\tilde{\mu}_n > \hat{\mu}_n$ at investor prior belief ρ_n.
Next, we show that Δ_0 is indeed differentiable with respect to μ by showing that both $V_U(\mu'(\mu,x))$ and $x_n(\mu)$ are differentiable with respect to μ almost everywhere. First, by Lebesgue's theorem for the differentiability of monotone functions, the monotonicity of $V_U(\cdot)$ implies that $V_U(\mu')$ is differentiable almost everywhere for $\mu' \in \mathcal{C}$. Second, on each interval $(x_n(\mu), x_{n+1}(\mu))$ such that the resulting reputation $\mu'(\mu,x) \in \mathcal{C}$, we have that

$$\rho' = \frac{(1-\lambda)\ell(x)\mu}{\ell(x)\mu + 1 - \mu} \tag{A.16}$$

is continuous and differentiable in both μ and x, and

$$\mu' = \rho' + (1-\rho')\sigma(\rho')$$

is continuously increasing in ρ' by the construction of intervals. Again, applying Lebesgue's theorem for the differentiability of monotone functions, we obtain that $\partial\mu'/\partial\rho'$ is well defined almost everywhere. To conclude, on each interval $(x_n(\mu), x_{n+1}(\mu))$ such that the resulting reputation $\mu'(\mu, x) \in \mathcal{C}$, we have that $V_U(\mu'(\mu, x))$ is differentiable with respect to μ almost everywhere, and

$$\frac{\partial V_U(\mu'(\mu, x))}{\partial \mu} = V_U'(\mu')\frac{\partial \mu'}{\partial \mu} = V_U'(\mu')\frac{\partial \mu'}{\partial \rho'}\frac{\partial \rho'}{\partial \mu}.$$

It then follows from equation (A.16) that

$$\frac{\partial \rho'}{\partial \mu} = \frac{(1-\lambda)\ell(x)}{(\ell(x)\mu + 1 - \mu)^2} \quad \text{and} \quad \frac{\partial \rho'}{\partial x} = \frac{(1-\lambda)\mu(1-\mu)\ell'(x)}{(\ell(x)\mu + 1 - \mu)^2}.$$

Using the fact that

$$\ell(x) = e^{\frac{2x-1}{2\phi^2}} \quad \text{and hence} \quad \ell'(x) = \ell(x)/\phi^2,$$

we have

$$\frac{\partial \rho'}{\partial \mu} = \frac{\phi^2 \partial \rho'/\partial x}{\mu(1-\mu)}. \tag{A.17}$$

Moreover, from equation,

$$\rho_n = \frac{(1-\lambda)\ell(x_n)\mu}{\ell(x_n)\mu + 1 - \mu},$$

we can solve $x_n(\mu)$ as

$$\ell(x_n) = \frac{(1-\mu)\rho_n}{\mu(1-\lambda-\rho_n)},$$

which implies that $x_n(\mu)$ is differentiable in μ and

$$\frac{\partial x_n(\mu)}{\partial \mu} = -\frac{\ell(x_n)}{\ell'(x_n)\mu(1-\mu)} = -\frac{\phi^2}{\mu(1-\mu)}. \tag{A.18}$$

Since Δ_0 is differentiable with respect to μ, we substitute equations (A.17) and (A.18) into equation (A.15) and get

$$0 = \frac{\phi^2}{\mu(1-\mu)}\left[\int_{\mu' \in \mathcal{C}} V_U'(\mu'(\mu, x))\frac{\partial \mu'}{\partial \rho'}\frac{\partial \rho'}{\partial x}\mathrm{d}\left[F^I(x) - F^U(x)\right]\right.$$
$$\left. + \sum_{n=1}^{N}(V_U(\tilde{\mu}_n) - V_U(\hat{\mu}_n))\left[f^I(x_n(\mu)) - f^U(x_n(\mu))\right]\right],$$

which implies

$$0 = \int_{\mu' \in \mathcal{C}} V_U'(\mu'(\mu, x)) \frac{\partial \mu'}{\partial x} \mathrm{d}[F^I(x) - F^U(x)] + \sum_{n=1}^{N} (V_U(\tilde{\mu}_n) - V_U(\hat{\mu}_n))[f^I(x_n(\mu)) - f^U(x_n(\mu))].$$

Notice that on each interval $(x_n(\mu), x_{n+1}(\mu))$ such that the resulting reputation $\mu' \notin \mathcal{C}$, we have that $V_U(\mu'(\mu, x))$ is a constant and hence $\mathrm{d}V_U(\mu'(\mu, x)) = 0$. Therefore, we can rewrite the above equation as

$$0 = \int_{-\infty}^{x_1(\mu)} [f^I(x) - f^U(x)] \mathrm{d}V_U(\mu'(\mu, x)) + \cdots + \int_{x_N(\mu)}^{\infty} [f^I(x) - f^U(x)] \mathrm{d}V_U (\mu'(\mu, x)) + \sum_{n=1}^{N} (V_U(\tilde{\mu}_n) - V_U(\hat{\mu}_n))[f^I(x_n(\mu)) - f^U(x_n(\mu))]. \tag{A.19}$$

Equation (A.19) uses the fact that $\mathrm{d}[F^I(x) - F^U(x)] = [f^I(x) - f^U(x)]\mathrm{d}x$ and $\mathrm{d}V_U(\mu'(\mu, x)) = V_U'(\mu'(\mu, x)) \frac{\partial \mu'}{\partial x} \mathrm{d}x$ for $\mu' \in \mathcal{C}$. Integration by parts cancels out the term $\sum_{n=1}^{N} (V_U(\tilde{\mu}_n) - V_U(\hat{\mu}_n))[f^I(x_n(\mu)) - f^U(x_n(\mu))]$, and hence yields

$$\int_{-\infty}^{\infty} V_U(\mu') \mathrm{d}[f^I(x) - f^U(x)] = -\frac{1}{\phi^2} \int_{-\infty}^{\infty} V_U(\mu') \Big[(x-1) f^I(x) - x f^U(x)\Big] \mathrm{d}x = 0,$$

which implies $\Delta_1(\mu) = 0$ for all $\mu \in \mathcal{I}$.

Step 2. We define

$$\Delta_n(\mu) \equiv \int_{-\infty}^{\infty} V_U(\mu'(\mu, x))[(x-1)^n f^I(x) - x^n f^U(x)]\mathrm{d}x$$

for $n = 1, 2, \ldots$, and show that $\Delta_n(\mu) = 0$ when n is odd and $\Delta_n(\mu)$ is a constant when n is even. This fact is established in Lemma A.1 below. □

LEMMA A.1: *For all $\mu \in \mathcal{I}$, $\Delta_n(\mu) = 0$ when n is odd and $\Delta_n(\mu)$ is a constant when n is even.*

PROOF OF LEMMA A.1: The lemma is proved by deduction. For $k = 0, 1, 2, \ldots$, we want to show that $\Delta_{2k}(\mu)$ is a constant and $\Delta_{2k+1}(\mu) = 0$, for all $\mu \in \mathcal{I}$. This is obviously true for $k = 0$ because we have established

$$\Delta_0(\mu) = \int_{-\infty}^{\infty} V_U(\mu')\Big[(x-1)^0 f^I(x) - x^0 f^U(x)\Big]\mathrm{d}x = \left[\frac{1}{\delta} - (1-\lambda)\right]c$$

and

$$\Delta_1(\mu) \equiv \int_{-\infty}^{\infty} V_U(\mu')\Big[(x-1)^1 f^I(x) - x^1 f^U(x)\Big]\mathrm{d}x = 0.$$

So, we only need to show that if $\Delta_{2k}(\mu)$ is a constant and $\Delta_{2k+1}(\mu) = 0$, for all $\mu \in \mathcal{I}$, then this is also true for $k+1$.

Since

$$\Delta_{2k+1}(\mu) = \int_{-\infty}^{\infty} V_U(\mu'(\mu, x))[(x-1)^{2k+1} f^I(x) - x^{2k+1} f^U(x)]\mathrm{d}x = 0$$

holds for all $\mu \in \mathcal{I}$, we have $\Delta'_{2k+1}(\mu) = 0$. Similar to the derivation in Step 2, we obtain

$$\int_{-\infty}^{\infty} V_U(\mu'(\mu, x))\mathrm{d}[(x-1)^{2k+1} f^I(x) - x^{2k+1} f^U(x)] = 0,$$

which implies that

$$\Delta_{2k+2}(\mu) = \int_{-\infty}^{\infty} V_U(\mu'(\mu, x))[(x-1)^{2k+2} f^I(x) - x^{2k+2} f^U(x)]\mathrm{d}x = \phi^2(2k+1)\Delta_{2k}(\mu).$$

Since $\Delta_{2k}(\mu)$ is a constant, $\Delta_{2k+2}(\mu)$ is also a constant for all $\mu \in \mathcal{I}$, which implies that $\Delta'_{2k+2}(\mu) = 0$. Hence, we have

$$\int_{-\infty}^{\infty} V_U(\mu'(\mu, x))\mathrm{d}[(x-1)^{2k+2} f^I(x) - x^{2k+2} f^U(x)] = 0,$$

which implies that

$$\Delta_{2k+3}(\mu) = \int_{-\infty}^{\infty} V_U(\mu'(\mu, x))[(x-1)^{2k+3} f^I(x) - x^{2k+3} f^U(x)]\mathrm{d}x = \phi^2(2k+2)\Delta_{2k+1}(\mu).$$

By deduction, we conclude that $\Delta_{2k}(\mu)$ is a constant and $\Delta_{2k+1}(\mu) = 0$, for all $\mu \in \mathcal{I}$. □

Step 3. Based on the finding that $\Delta_n(\mu) = 0$ whenever n is odd, we argue that

$$V_U(\mu'(\mu, x)) + V_U(\mu'(\mu, 1 - x)) \tag{A.20}$$

is constant across $\mu \in \mathcal{I}$ and x, which we formally show in Lemma A.2 below.

LEMMA A.2: *If $\Delta_n(\mu) = 0$ whenever n is odd, then we must have*

$$V_U(\mu'(\mu, x)) + V_U(\mu'(\mu, 1 - x)) = C$$

for any $\mu \in \mathcal{I}$, where C is a constant independent of x and μ.

PROOF OF LEMMA A.2: First, by defining $y = x - 1$, we have

$$\int_{-\infty}^{\infty} V_U(\mu'(\mu, x))(x - 1)^n f^I(x)\mathrm{d}x = \int_{-\infty}^{\infty} V_U(\mu'(\mu, y + 1))y^n f^U(y)\mathrm{d}y.$$

Hence,

$$\int_{-\infty}^{\infty} V_U(\mu'(\mu, x))[(x - 1)^n f^I(x) - x^n f^U(x)]\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \left[V_U(\mu'(\mu, x + 1)) - V_U(\mu'(\mu, x))\right]x^n f^U(x)\mathrm{d}x.$$

Define $h(x; \mu) = V_U(\mu'(\mu, x + 1)) - V_U(\mu'(\mu, x))$. We want to show that $h(x; \mu)$ is even in x for any μ. We can always decompose $h(x; \mu)$ into

$$h(x; \mu) = \frac{h(x; \mu) + h(-x; \mu)}{2} + \frac{h(x; \mu) - h(-x; \mu)}{2}.$$

It is then easy to check that $h_1(x; \mu) \triangleq \frac{h(x;\mu)+h(-x;\mu)}{2}$ is even in x, while $h_2(x; \mu) \triangleq \frac{h(x;\mu)-h(-x;\mu)}{2}$ is odd in x. When n is odd,

$$\int_{-\infty}^{\infty} h_1(x; \mu)x^n f^U(x)\mathrm{d}x = 0,$$

as $h_1(x; \mu)x^n$ is an odd function in x. Thus, the assumption that $\Delta_n(\mu) = 0$ whenever n is odd implies that

$$\int_{-\infty}^{\infty} h_2(x; \mu)x^n f^U(x)\mathrm{d}x = 0$$

when n is odd. Moreover, since $h_2(x;\mu)$ is odd, $h_2(x;\mu)x^n$ is also odd when n is even. We therefore get

$$\int_{-\infty}^{\infty} h_2(x;\mu)x^n f^U(x)\mathrm{d}x = 0$$

for any $n = 0, 1, 2, \ldots$. Because the value function V_U is bounded, $|h_2|$ is also bounded away from ∞. Lemma A.3 below shows that $h_2(x;\mu) = 0$ for each x, which implies that $h(x;\mu) - h(-x;\mu) = 0$.

Substituting this into the expression of h yields

$$V_U(\mu'(\mu, x+1)) - V_U(\mu'(\mu, x)) = V_U(\mu'(\mu, -x+1)) - V_U(\mu'(\mu, -x)),$$

and hence

$$V_U(\mu'(\mu, x)) + V_U(\mu'(\mu, 1-x)) = V_U(\mu'(\mu, x+1)) + V_U(\mu'(\mu, -x)) \tag{A.21}$$

for any $\mu \in \mathcal{I}$. Define $\xi(x;\mu) \triangleq V_U(\mu'(\mu, x)) + V_U(\mu'(\mu, 1-x))$. From equation (A.21), it is easy to see that $\xi(x;\mu)$ is an even function and thus is symmetric with respect to zero. We also note that $\xi(x;\mu)$ is symmetric with respect to 1/2. So, $\xi(x;\mu)$ must be a periodic function in x with period 1. It therefore follows from equation (A.21) that

$$\begin{aligned} & V_U(\mu'(\mu, x)) + V_U(\mu'(\mu, 1-x)) \\ & = V_U(\mu'(\mu, x+k)) + V_U(\mu'(\mu, 1-x-k)) \\ & = V_U(\mu'(\mu, x+k+1)) + V_U(\mu'(\mu, -k-x)). \end{aligned} \tag{A.22}$$

Moreover, since equation (A.22) is true for all $x \in (-\infty, \infty)$ and $k \in \mathbb{N}$, we can let $k \to \infty$ for any fixed x. In this case, $\rho'(\mu, x+k+1) \to 1-\lambda$ and $\rho'(\mu, -k-x) \to 0$ for all $\mu \in \mathcal{I}$. Denote by $\underline{\mu}$ the limiting reputation when the investor prior belief goes to 0 and by $\bar{\mu}$ the limiting reputation when the prior belief goes to $1-\lambda$. We then have

$$V_U(\mu'(\mu, x)) + V_U(\mu'(\mu, 1-x)) = V_U(\underline{\mu}) + V_U(\bar{\mu}),$$

where the right-hand side of the above equation, $V_U(\underline{\mu}) + V_U(\bar{\mu})$, is a constant independent of x and μ. □

Step 4. We are now able to derive the contradiction.

We know that if $\mu \in \mathcal{I}$ is strictly increasing in ρ, there must exist an interval of ρ, $\hat{\mathcal{I}} \subset [0, 1-\lambda]$, such that $\mu = \rho + \sigma(\rho)(1-\rho) \in \mathcal{I}$ for $\rho \in \hat{\mathcal{I}}$. Given any $\mu \in \mathcal{I}$ and signal realization $x \in (-\infty, \infty)$, the equilibrium belief updating system (4) to (6) implies that the resulting investor prior belief in the next period ρ' increases from 0 to $1-\lambda$ as x increases from $-\infty$ to ∞. Therefore, there exists a signal realization x that leads to $\rho' \in \hat{\mathcal{I}}$, and hence, the reputation in the next period satisfies $\mu' \in \mathcal{I}$. Fix the signal realization to x. Then a marginal increase in μ will lead to a marginal increase in $\mu'(\mu, x)$, since this marginal increase leads to a higher $\rho' \in \hat{\mathcal{I}}$, and μ' is strictly increasing in ρ' by construction. As $\mu'(\mu, x)$ strictly increases and $\mu'(\mu, 1-x)$ is nondecreasing when μ increases marginally, we conclude that $V_U(\mu'(\mu, x)) + V_U(\mu'(\mu, 1-x))$ is strictly increasing in μ from the fact that $V_U(\mu)$ is strictly increasing. However, this contradicts the requirement that $V_U(\mu'(\mu, x)) + V_U(\mu'(\mu, 1-x)) = C$, which is established in equation (A.20).

LEMMA A.3: *Consider a bounded odd function $g(x)$ that, for any $n = 0, 1, 2, \ldots,$ satisfies*

$$\int_{-\infty}^{\infty} g(x)x^n f^U(x)\mathrm{d}x = 0.$$

Then we must have $g(x) = 0$ almost everywhere.

PROOF OF LEMMA A.3: Suppose by contradiction that $g(x) \neq 0$ for a nonnegative measure of x. Then this implies that $\int_{-\infty}^{\infty} |g(x)| f^U(x)\mathrm{d}x > 0$.

Let

$$f_+(x) = \frac{2g_+(x)f^U(x)}{\int_{-\infty}^{\infty} |g(x)| f^U(x)\mathrm{d}x} = \frac{2\max\{g(x), 0\} f^U(x)}{\int_{-\infty}^{\infty} |g(x)| f^U(x)\mathrm{d}x},$$

and

$$f_-(x) = \frac{2g_-(x)f^U(x)}{\int_{-\infty}^{\infty} |g(x)| f^U(x)\mathrm{d}x} = \frac{2\max\{-g(x), 0\} f_U(x)}{\int_{-\infty}^{\infty} |g(x)| f^U(x)\mathrm{d}x}.$$

As an odd function, g satisfies

$$\int_{-\infty}^{\infty} g(x) f^U(x)\mathrm{d}x = 0,$$

which implies that

$$\int_{-\infty}^{\infty} g_+(x) f^U(x)\mathrm{d}x = \int_{-\infty}^{\infty} g_-(x) f^U(x)\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} |g(x)| f^U(x)\mathrm{d}x.$$

So, both $f_+(x)$ and $f_-(x)$ are probability densities. Let x_+ and x_- denote the random variables with densities $f_+(x)$ and $f_-(x)$, respectively.

Moreover, since equation

$$\int_{-\infty}^{\infty} g(x)x^n f^U(x)\mathrm{d}x = 0$$

holds for any $n = 0, 1, 2, \ldots$, we must have

$$\int_{g(x)\geq 0} g(x)x^n f^U(x)\mathrm{d}x = -\int_{g(x)\leq 0} g(x)x^n f^U(x)\mathrm{d}x,$$

which implies that

$$\mathbb{E}X_+^n = \mathbb{E}X_-^n$$

holds for all $n = 0, 1, 2, \cdots$. We also have that for any $t \in \mathbb{R}$, both $\mathbb{E}e^{t|X_+|}$ and $\mathbb{E}e^{t|X_-|}$ are bounded away from ∞, because they are both lower than $\int_{-\infty}^{\infty} e^{t|x|}|g(x)| f^U(x)\mathrm{d}x$, which is strictly less than ∞ as g is bounded. Then, Lemma A.4 below immediately implies that random variables x_+ and x_- should have the same density functions, which can happen only when $g(x) = 0$ almost everywhere from the definition of $f_+(x)$ and $f_-(x)$. □

LEMMA A.4: *Consider two random variables X and Y that, for any $n = 0, 1, 2, \ldots$, satisfy*

$$\mathbb{E}X^n = \mathbb{E}Y^n,$$

and both $\mathbb{E}e^{t|X|}$ and $\mathbb{E}e^{t|Y|}$ are bounded away from ∞ for any $t \in \mathbb{R}$. Then the distribution functions of X and Y are the same almost everywhere.

PROOF OF LEMMA A.4: By definition, we have that, for any x,

$$e^{tx} = \sum_{k=0}^{\infty} \frac{t^k x^k}{k!}$$

and

$$e^{tx} \leq \sum_{k=0}^{\infty} \frac{t^k |x|^k}{k!} = e^{t|x|}.$$

s $\mathbb{E}e^{t|X|} < \infty$, we can apply the dominated convergence theorem to get

$$\mathbb{E}e^{tX} = \sum_{k=0}^{\infty} \frac{t^k \mathbb{E}X^k}{k!}.$$

Similarly, we have

$$\mathbb{E}e^{tY} = \sum_{k=0}^{\infty} \frac{t^k \mathbb{E}Y^k}{k!}.$$

Because

$$\mathbb{E}X^k = \mathbb{E}Y^k$$

for any $k = 0, 1, 2, \ldots$, we obtain

$$\mathbb{E}e^{tX} = \mathbb{E}e^{tY}$$

for all $t \in \mathbb{R}$. Since the two random variables X and Y always have the same moment-generating functions, it follows that they must have the same distribution functions almost everywhere. □

PROOF OF PROPOSITION 4: The proof is similar to that of Proposition 3. Suppose that there exists an open interval of equilibrium reputations $\mathcal{I} \subset \mathcal{U}$ such that the fund is indifferent between acquiring information and remaining uninformed. Then the indifference condition implies that for a continuum of $\mu \in \mathcal{I}$, $V_I(\mu) - V_U(\mu) = c$, which implies that

$$\Delta_0(\mu) \triangleq \int_{-\infty}^{\infty} \Psi(\mu')\mathrm{d}F^I(x) + \int_{-\infty}^{\infty} V_U(\mu')\mathrm{d}[F^I(x) - F^U(x)] = \frac{c}{\delta},$$

where

$$\Psi(\mu') = (1-\lambda)\big[V_I(\rho') - V_U(\rho')\big] < (1-\lambda)c$$

if the fund strictly prefers not to acquire information, and $\Psi(\mu') = (1-\lambda)c$ otherwise.

Since μ' can be discontinuous in ρ' for at most countably many prior beliefs $\{\rho_n\}_{n=1}^N$, with $\rho_1 < \cdots < \rho_N$ and $N \in \{0, 1, \ldots, \infty\}$ being the total number of discontinuous point, there exist at most countably many investment outcomes $\{x_n(\mu)\}_{n=1}^N$ with $x_1(\mu) < \cdots < x_N(\mu)$ satisfying $\rho_n = \frac{(1-\lambda)\ell(x_n)\mu}{\ell(x_n)\mu+1-\mu}$ such that the reputations μ' are discontinuous in x at these investment outcomes. We can thus rewrite $\Delta_0(\mu)$ as

$$\begin{aligned}\Delta_0(\mu) \triangleq & \int_{-\infty}^{x_1(\mu)} V_U(\mu')\mathrm{d}[F^I(x) - F^U(x)] + \cdots + \int_{x_N(\mu)}^{\infty} V_U(\mu')\mathrm{d}[F^I(x) - F^U(x)] \\ & + \int_{-\infty}^{x_1(\mu)} \Psi(\mu')\mathrm{d}F^I(x) + \cdots + \int_{x_N(\mu)}^{\infty} \Psi(\mu')\mathrm{d}F^I(x).\end{aligned}$$

We again apply the standard arguments in Stokey and Lucas (1989) and obtain that both V_I and V_U are strictly increasing in $\mathcal{U}$. On each interval $(x_n(\mu), x_{n+1}(\mu))$ for $n = 0, \cdots, N$ with $x_0(\mu) = -\infty$ and $x_{N+1}(\mu) = \infty$, one of two cases must occur for both $V_U(\mu')$ and $\Psi(\mu')$: either $V_U(\mu')/\Psi(\mu')$ is a constant, or it is changing with x. When the function $V_U(\mu')$ or $\Psi(\mu')$ is changing with x, it is differentiable almost everywhere by the monotonicity of both the V_I and V_U functions. Also, the cutoff investment outcomes are differentiable in μ as well. Hence, Δ_0 is differentiable with respect to μ. Then, similar to the proof of Proposition 3, by deduction and by integrating by parts, we get

$$0 = \int_{-\infty}^{\infty} \Psi(\mu')(x-1)^n f^I(x)\mathrm{d}x + \int_{-\infty}^{\infty} V_U(\mu')[(x-1)^n f^I(x) - x^n f^U(x)]\mathrm{d}x$$

whenever n is odd.

The above equation can be rewritten as

$$0 = \int_{-\infty}^{\infty} h(x, \mu) x^n f^U(x) \mathrm{d}x,$$

where

$$h(x, \mu) = \Psi(\mu'(\mu, x + 1)) + V_U(\mu'(\mu, x + 1)) - V_U(\mu'(\mu, x)).$$

The argument in Proposition 3 then implies that

$$0 = \int_{-\infty}^{\infty} [h(x, \mu) - h(-x, \mu)] x^n f^U(x) \mathrm{d}x,$$

for $n = 0, 1, 2, \ldots$. Applying Lemma A.3, we get for each $\mu \in \mathcal{I}$ and any $x \in \mathbb{R}$,

$$\begin{aligned} &V_U(\mu'(\mu, x)) + V_U(\mu'(\mu, 1 - x)) + \Psi(\mu'(\mu, 1 - x)) \\ &= V_U(\mu'(\mu, -x)) + V_U(\mu'(\mu, 1 + x)) + \Psi(\mu'(\mu, 1 + x)). \end{aligned} \tag{A.23}$$

Let $x \to \infty$. Then both $\rho'(\mu, x)$ and $\rho'(\mu, 1 + x)$ converge to $1 - \lambda$, while both $\rho'(\mu, -x)$ and $\rho'(\mu, 1 - x)$ converge to zero, for all $\mu \in \mathcal{I}$. Therefore, it is always the case that $V_U(\mu'(\mu, x)) + V_U(\mu'(\mu, 1 - x)) = V_U(\mu'(\mu, -x)) + V_U(\mu'(\mu, 1 + x))$ as $x \to \infty$. So, equation (A.23) implies that

$$\lim_{x\to\infty} \Psi(\mu'(\mu, 1 - x)) = \lim_{x\to\infty} \Psi(\mu'(\mu, 1 + x)).$$

Notice that if the fund does not acquire information when the prior belief is close to 0, $\lim_{x\to\infty} \mu'(\mu, 1 - x) = \lim_{x\to\infty} \rho'(\mu, 1 - x) = 0$. Hence, on the one hand, $\lim_{x\to\infty} \Psi(\mu'(\mu, 1 - x)) = 0$ if the fund does not acquire information when the prior belief is close to 0, and $(1 - \lambda)c$ otherwise. On the other hand, as shown by the proof of Lemma 3 below, $\lim_{x\to\infty} \Psi(\mu'(\mu, 1 + x))$ is bounded away from zero if the fund does not acquire information when the prior belief is close to $1 - \lambda$, and is $(1 - \lambda)c$ otherwise. Therefore, as long as the fund does not acquire information when the prior belief is close to either 0 or $1 - \lambda$, $\lim_{x\to\infty} \Psi(\mu'(\mu, 1 - x)) \neq \lim_{x\to\infty} \Psi(\mu'(\mu, 1 + x))$, which contradicts equation (A.23). □

PROOF OF LEMMA 3: Suppose that there is a $\bar{\rho} \in [0, 1 - \lambda]$ such that the fund with prior belief $\rho \in [\bar{\rho}, 1 - \lambda]$ chooses not to acquire information for sure in equilibrium.

Then, for $\rho \in [\bar{\rho}, 1-\lambda]$, $\mu(\rho) = \rho$, and hence,

$$
\begin{aligned}
W_I(\rho) - W_U(\rho) &= V_I(\rho) - V_U(\rho) \\
&= \delta \int_{-\infty}^{\infty} \big[(1-\lambda)V_I(\mu') + \lambda V_U(\mu')\big] \mathrm{d}F^I(x) - \delta \int_{-\infty}^{\infty} V_U(\mu')\mathrm{d}F^U(x) \\
&= \delta \int_{-\infty}^{\infty} \Psi(\mu')\mathrm{d}F^I(x) + \delta \int_{-\infty}^{\infty} V_U(\mu')\mathrm{d}[F^I(x) - F^U(x)],
\end{aligned}
$$

where

$$\Psi(\mu') = (1-\lambda)\big[V_I(\mu') - V_U(\mu')\big].$$

The above equation can be rewritten as

$$
\begin{aligned}
V_I(\rho) - V_U(\rho) = \delta \int_{-\infty}^{\infty} \Psi(\mu')\ell(x) f^U(x)\mathrm{d}x \\
+ \delta \int_{-\infty}^{\infty} \big[V_U(\mu'(\rho, 1+x)) - V_U(\mu'(\rho, x))\big] f^U(x)\mathrm{d}x.
\end{aligned}
$$

The first part of the above equality comes from the fact that $f^I(x) = \ell(x) f^U(x)$, and the second part comes from changing variables by letting $y = x-1$ for $f^I(x)$ (similar to the proof of Lemma A.2). Notice that there are two cases at $\rho'(\rho, x)$: either $\mu'(\rho, x) > \rho'(\rho, x)$ or $\mu'(\rho, x) = \rho'(\rho, x)$. If $\mu'(\rho, x) > \rho'(\rho, x)$, then the uninformed fund acquires information with positive probability and hence $\Psi(\mu') = (1-\lambda)c$. If $\mu'(\rho, x) = \rho'(\rho, x) = \frac{(1-\lambda)\ell(x)\mu}{\ell(x)\mu+1-\mu}$, then we must have that $\mu'(\rho, 1+x) > \mu'(\rho, x)$ as $\mu'(\rho, 1+x) \geq \rho'(\rho, 1+x) > \rho'(\rho, x)$, which implies that

$$V_U(\mu'(\rho, 1+x)) - V_U(\mu'(\rho, x)) \geq \pi\theta(\rho'(\rho, 1+x)) - \pi\theta(\rho'(\rho, x)).$$

Therefore, we obtain

$$
\begin{aligned}
&\Psi(\mu')\ell(x) + \big[V_U(\mu'(\rho, 1+x)) - V_U(\mu'(\rho, x))\big] \\
&\geq \min\{(1-\lambda)c\ell(x), \pi\theta(\rho'(\rho, 1+x)) - \pi\theta(\rho'(\rho, x))\} \\
&\triangleq \Phi(x; \rho).
\end{aligned}
$$

For $\rho \in [\bar{\rho}, 1-\lambda]$, define

$$\Delta(\rho) = \delta \int_{-\infty}^{\infty} \Phi(x; \rho) f^U(x)\mathrm{d}x.$$

Obviously, $\Delta(1-\lambda) > 0$ as $\Phi(x; 1-\lambda) > 0$ almost everywhere. So, we can just define $\bar{c} = \Delta(1-\lambda)$. From the continuity of $\Delta(\rho)$, then if $c < \bar{c}$, there must exist some ρ sufficiently close to $1-\lambda$ such that $\Delta(\rho) > c$, which implies that the uninformed fund with prior belief ρ prefers to acquire information. This contradicts the assumption that the uninformed fund with prior belief $\rho \in [\bar{\rho}, 1-\lambda]$ strictly prefers not to acquire information in equilibrium.

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Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's website:

Appendix S1: Internet Appendix.
Replication code.